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Estimates of Critical Lengths and Critical Temperatures for Classical and Quantum Lattice Systems

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Local Ward identities are derived which lead to the mean-field upper bound for the critical temperature for certain multicomponent classical lattice systems (improving by a factor of two an estimate of Brascamp–Lieb). We develop a method for accurately estimating lattice Green's functions I_d yielding $0.3069 < I_4 < 0.3111$ and the global bounds $(d - \frac{1}{2})^{-1} < I_d < (d - 1)^{-1}$ for all $d \geqslant 4$. The estimate for I_d implies the existence of a critical length for classical lattice systems with fixed length spins. For ν-component spins with fixed length b on the lattice $\mathbb{Z}^d$, $\nu = 1, 2, 3, 4$, the critical temperature for spontaneous magnetization satisfies

$$\frac{2Jb^2}{k}\frac{d-1}{\nu} < T_c \leqslant \frac{2Jb^2}{k}\frac{d}{\nu} \qquad \text{for } d \geqslant 4$$

Using GHS or generalized Griffiths' inequalities, we find that the upper bounds on the critical temperature extend to certain classical and quantum systems with unbounded spins. Absence of symmetry breakdown at high temperature for quantum lattice fields follows from bounding the energy density by a multiple of kT. Path space techniques for finite degrees of freedom show that the high-temperature limit is classical.

KEY WORDS: Mean field; Ward identities; path space; Green's functions.

1. INTRODUCTION

Certain quantum systems at nonzero temperature T may be associated with a (continuous) classical system (path space) with a finite length $\beta = 1/kT$ in one direction (the β direction). This "temperature direction" corresponds to the "imaginary time direction" of Euclidean field theory.[16,21] The

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existence of a critical temperature for the quantum system corresponds to the existence of a critical length for the classical system. As a result of this correspondence, analogous techniques may be used to study this critical behavior of classical and quantum systems. We derive a mean field bound (Griffiths' third inequality[17]) for the absence of spontaneous magnetization for one-component classical lattice systems, which may be applied to quantum lattice fields by using a GHS inequality[13,38] for the Duhamel correlation functions,[6] together with an equipartition bound (Sections 6–8). For two-, three-, and four-component classical lattice systems we use a local Ward identity (Section 3) and generalized Griffiths' inequalities[1,7] to obtain the mean-field bound. This mean-field bound improves by a factor of two an estimate of Brascamp and Lieb.[4] For fixed-length rotators with arbitrarily many components we obtain a bound which is almost the mean-field bound, and which converges to the mean-field bound as the number of components increases.

The mean-field bound provides a good upper bound on the critical temperature of classical lattice systems. The lower bound derived from the infrared bound[11] is estimated by an accurate computation of lattice Green's functions (Section 5). These estimates prove the existence of a critical length for the breakdown of a (continuous) symmetry. Critical lengths for certain continuum classical systems follow from the existence of a critical temperature for the associated quantum system.

Quantum field models in two space–time dimensions do not have long-range order at nonzero temperature. This follows from Nelson's symmetry,[29,36] applying the transfer matrix in the space direction.[21] On the other hand, the Peierls' argument shows that at zero temperature the ground state (vacuum) is degenerate for one-component models.[18] However, even at zero temperature there are no Goldstone bosons[3b,5,12] and so there is no continuous symmetry breaking. In fact, for two-component models the vacuum is unique.[3b]

In three space–time dimensions both discrete and continuous symmetries can be broken in the ground state.[11] On the other hand, there are no Goldstone bosons in three space–time dimensions at nonzero temperature.[20] Thus a continuous symmetry cannot be broken at nonzero temperature.

In four space–time dimensions the existence of a critical temperature for the breakdown of a (continuous) symmetry for quantum lattice fields follows from the control of both the low-temperature and high-temperature limits. The breakdown of symmetry at low temperature follows from infrared bounds[11] as discussed in Ref. 10 (quantum crystals). We briefly review in Section 9 the proof of symmetry breaking at nonzero temperature, based on infrared bounds in four space–time dimensions and on the Peierls' argument in three space–time dimensions (discrete symmetry). To show the absence of symmetry breakdown at high temperatures (and ergodicity of the KMS state in certain cases using correlation inequalities as in Corollary 3.3 and Refs. 3a and 3b), we use local Ward identities and either GHS or generalized Griffiths' inequalities. To apply this method, one must control the high-temperature divergences, which amounts to obtaining an equipartition bound: $(1/V)\langle H\rangle_\beta \leqslant ckT$, for some constant c, where H is the Hamiltonian, V the volume of the system, and T the temperature. This is done by reducing the estimate to one involving only one degree of freedom and then showing that the high-temperature limit is classical—which follows by path space techniques (Sections 6–8).

The unifying aspect of the various results obtained is the correspondence between a quantum system with inverse temperature β and the associated classical system with length β. As a further illustration of this correspondence we consider a simple quantum system (Ising model with transverse external field) for which the associated classical system is a continuum Ising model (Sections 2.5 and 10). The particular functional form of the dependence of spontaneous magnetization and critical temperature on the Ising couplings (see Ref. 9) is such that these quantities converge in the continuum limit determining the path space of the quantum system.

2. CLASSICAL AND QUANTUM SYSTEMS

In this section we describe the models to be considered and we discuss some of their general properties which will be useful in later sections.

2.1. Classical Systems

Let Λ be a finite subset of $\mathbb{Z}^d$. To each point $i \in \Lambda$ is associated a spin $\sigma_i \in \mathbb{R}^\nu$ (ν-component spin) with a priori single site distribution $d\mu_0(\sigma_i)$, which depends only on $\|\sigma_i\| = (\sum_{\alpha=1}^{\nu} \sigma_i^{(\alpha)2})^{1/2}$, $\sigma_i^{(\alpha)}$ being the α component of σ_i.

The interaction is a ferromagnetic pair interaction of the form

$$H = \frac{1}{2} \sum_{\langle ij\rangle\in\Lambda} J_{ij}\|\sigma_i - \sigma_j\|^2 - h \sum_{i\in\Lambda} \sigma_i^{(1)}$$

where $J_{ij} \geqslant 0$, $h \geqslant 0$, and $\langle ij\rangle$ denotes pairs of points in $\mathbb{Z}^d$ (each pair counted only once). We define $\sigma_A^{(\alpha)} = \prod_{i\in A} \sigma_i^{(\alpha)}$, where A is a subset of Λ with finite *multiplicity*. That is, the set A may contain a given point of Λ more than once. For example, $\sigma_3^{(1)2}\sigma_4^{(1)} = \sigma_A^{(1)}$, where $A = \{3, 3, 4\}$.

Denoting by $\langle\cdots\rangle$ the expectation value at inverse temperature β,

$$\langle\sigma_A\rangle = \frac{\int \sigma_A e^{-\beta H} \prod_i d\mu_0(\sigma_i)}{\int e^{-\beta H} \prod_i d\mu_0(\sigma_i)}$$

the following properties of these models are known:

(a) $\nu = 1$, μ_0 arbitrary (even); Griffiths' inequalities (e.g., Refs. 15 and 39)

$$\langle\sigma_A\sigma_B\rangle \geqslant \langle\sigma_A\rangle\langle\sigma_B\rangle \geqslant 0$$

(b) $\nu = 1$, $d\mu_0 = e^{-V(|\sigma|)}\,d\sigma$, $V'(\xi)$ convex for $\xi \geqslant 0$; GHS inequality[8,13,39]

$$\begin{aligned} 0 \geqslant \langle\sigma_i\sigma_j\sigma_k\rangle^T = \langle\sigma_i\sigma_j\sigma_k\rangle &- \langle\sigma_i\rangle\langle\sigma_j\sigma_k\rangle - \langle\sigma_j\rangle\langle\sigma_i\sigma_k\rangle \\ &- \langle\sigma_k\rangle\langle\sigma_i\sigma_j\rangle + 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle \end{aligned}$$

(c) $\nu = 2$; generalized Griffiths' inequalities[1,7,43]

$$\langle\sigma_{\mathrm{A}}^{(1)}\sigma_{\mathrm{B}}^{(2)}\rangle \geqslant 0$$

If $d\mu_0 = \exp[-V(\|\sigma\|^2)]\,d\sigma^{(1)}\,d\sigma^{(2)}$, $V(\xi)$ convex,

$$\langle\sigma_A^{(\alpha)}\sigma_B^{(\alpha)}\rangle \geqslant \langle\sigma_A^{(\alpha)}\rangle\langle\sigma_B^{(\alpha)}\rangle, \qquad \alpha = 1, 2$$

$$\langle\sigma_A^{(1)}\sigma_B^{(2)}\rangle \leqslant \langle\sigma_A^{(1)}\rangle\langle\sigma_B^{(2)}\rangle$$

In Appendix A.1 we prove inequality (c), which is stronger than the versions appearing in the literature. [The same method gives a straightforward proof of (b).]

(d) $\nu = 3, 4$; $d\mu_0 = \exp[-a(\|\sigma\|^2)^2 + b\|\sigma\|^2]\prod_{i=1}^{\nu} d\sigma_i$ or $\delta(\|\sigma\|^2 - R^2)$.[7] Inequalities (c) hold for any *two* among the ν-components.

Remark. By considering limits of measures in (b) and (c), it follows that the correlation inequalities hold also for fixed length $\delta(\|\sigma\|^2 - R^2)$ and uniform $\theta(R^2 - \|\sigma\|^2)\,d\sigma^{(1)}\,d\sigma^{(2)}$, $\theta(R^2 - \sigma^2)\,d\sigma$ distributions.

2.2. Quantum Systems

Consider a quantum system with Hamiltonian H, algebra of observables $\mathcal{O}$, and an abelian subalgebra $\mathscr{A} \subset \mathcal{O}$. The equilibrium state at inverse temperature β is given by

$$\langle A\rangle = \mathrm{Tr}(e^{-\beta H}A)/\mathrm{Tr}\, e^{-\beta H} \qquad \text{for } A \in \mathcal{O} \tag{1}$$

(supposing $e^{-\beta H}$ is trace-class).

If the kernel of $e^{-\beta H}$ is nonnegative, a "path space" may be constructed so that the expectations of elements of $\mathscr{A}$ may be calculated by classical function space techniques.[14,16,19,21,24,25] The general relation is

$$\frac{\operatorname{Tr}(e^{-(\beta-t_n)H}A_n e^{-(t_n-t_{n-1})H}A_{n-1}\cdots e^{-(t_2-t_1)H}A_1 e^{-t_1 H})}{\operatorname{Tr} e^{-\beta H}} = \langle \hat{A}_n(t_n)\cdots \hat{A}_1(t_1)\rangle$$

$$0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant \beta \tag{2}$$

where $\hat{A}$ denotes a classical random variable [$\mathscr{A}(t) = \{\hat{A}(t), A \in \mathscr{A}\}$ is isomorphic to $\mathscr{A}$]. The positivity of the kernel of $e^{-\beta H}$ is expressed by the fact that the left-hand side of (2) is nonnegative if $A_1,\ldots, A_n \geqslant 0$. If the quantum system has s space dimensions, the associated classical system has $d = s + 1$ dimensions, the additional dimension being associated with the parameter t. This dimension will be called the "β direction."

The Hamiltonian generates an automorphism of $\mathscr{O}$ given by

$$B_\alpha = e^{i\alpha H}Be^{-i\alpha H}$$

The KMS condition, following from (1), is

$$\langle B_{-i\beta}C\rangle = \langle CB\rangle \tag{3}$$

Expressed in terms of the path space (2), with $B, C \in \mathscr{A}$, (3) becomes

$$\langle \hat{B}(\beta)\hat{C}(0)\rangle = \langle \hat{C}(0)\hat{B}(0)\rangle = \langle \hat{B}(0)\hat{C}(0)\rangle = \langle \hat{B}(\beta)\hat{C}(\beta)\rangle \tag{4}$$

where the second equality in (4) follows since $\mathscr{A}$ is abelian. Equation (4) implies $\hat{B}(\beta) = \hat{B}(0)$; i.e., the path space is *periodic* with period β. Thus in the β direction the length is β with periodic boundary conditions.[10,16,21]

If the Hamiltonian depends on a parameter λ, we use the notation

$$H = H(\lambda = 0), \qquad H' = (d/d\lambda)|_{\lambda=0}H(\lambda)$$

$$\langle B\rangle_\lambda = \operatorname{Tr}(e^{-\beta H(\lambda)}B)/\operatorname{Tr} e^{-\beta H(\lambda)}, \qquad \langle B\rangle = \langle B\rangle_0$$

We have then

$$(d/d\lambda)|_{\lambda=0}\langle B\rangle_\lambda = -\beta[(H', B) - \langle H'\rangle\langle B\rangle] \tag{5}$$

where the Duhamel two-point function (A, B) is defined by (see Ref. 6 and references therein)

$$(A, B) = \int_0^1 dx \operatorname{Tr}(e^{-x\beta H}Ae^{-(1-x)\beta H}B)/\operatorname{Tr} e^{-\beta H} = (B, A) \tag{6}$$

Therefore

$$(A^*, A) = \int_0^1 dx\, D(x)$$

where

$$D(x) = \mathrm{Tr}[(e^{-x\beta H/2}A^* e^{-(1-x)\beta H/2})(e^{-(1-x)\beta H/2}Ae^{-x\beta H/2})]/\mathrm{Tr}\, e^{-\beta H}$$

is

(a) *Nonnegative*, leading to the Schwartz inequality

$$|(A, B)|^2 \leqslant (A, A^*)(B^*, B) \tag{7}$$

(b) *Convex*, leading to the bound

$$(A^*, A) \leqslant \tfrac{1}{2}\langle AA^* + A^*A\rangle \tag{8}$$

since

$$\int_0^1 dx\, D(x) \leqslant \tfrac{1}{2}[D(1) + D(0)] = \tfrac{1}{2}\langle A^*A + AA^*\rangle$$

It is important to notice that if $A, B \in \mathscr{A}$ the Duhamel two-point function may be expressed as an average in the β direction of path space expectations:

$$(A, B) = \int_0^1 dx \langle \hat{A}(\beta x)\hat{B}(0)\rangle \tag{9}$$

We may thus apply correlation inequalities to (A, B) if they are valid for path space expectations. The Duhamel function is convenient when discussing quantum systems, although all results could be formulated directly in terms of the path space.

These general ideas are illustrated by the following models.

2.3. Schrödinger Particle

The algebra $\mathscr{A}$ is made up of functions of the position operator q and the Hamiltonian has the form

$$H = \tfrac{1}{2}\pi^2 + V(q)$$

In Appendix B we give a derivation of the corresponding path space. For related discussions see Refs. 19b and 21. We obtain

$$\begin{aligned}\mathrm{Tr}\exp\{-\beta[\tfrac{1}{2}\pi^2 + V(q)]\}\\ = (2\pi\beta)^{-1/2}\int_{-\infty}^{+\infty} dx \left\langle \exp\left[-\beta\int_0^1 dt\, V(x + \sqrt{\beta}r(t))\right]\right\rangle_r\end{aligned} \tag{10}$$

where $r(t)$ is the Gaussian stochastic process with covariance

$$\begin{aligned}\langle r(t_1)r(t_2)\rangle_r &= \sum_{n\neq 0}\frac{e^{i2\pi n(t_1-t_2)}}{(2\pi n)^2}, \qquad t_1, t_2 \in [0, 1]\\ &= \tfrac{1}{2}\{(t_1 - t_2)^2 - |t_1 - t_2| + \tfrac{1}{6}\} \qquad \text{if} \quad -\tfrac{1}{2} \leqslant t_1 - t_2 \leqslant \tfrac{1}{2}\end{aligned} \tag{11}$$

Expectations are computed by writing $q(t) = q_0 + \sqrt{\beta}r(t/\beta)$, where dq_0 is $(2\pi\beta)^{-1/2} \times$ Lebesgue measure. Note that this path space is translation invariant $(t \to t + t_0)$.

2.4. Quantum Lattice Fields

To each point $x \in \Lambda$, a rectangular sublattice of $\mathbb{Z}^s$, are associated ν Schrödinger particles with positions $\boldsymbol{\phi}(x) = (q^{(1)}(x),..., q^{(\nu)}(x))$ and momenta $\boldsymbol{\pi}(x) = (\pi^{(1)}(x),..., \pi^{(\nu)}(x))$. The Hamiltonian H has the form

$$H = \sum_{x\in\Lambda} [\tfrac{1}{2}\boldsymbol{\pi}(x)^2 + V(\boldsymbol{\phi}(x))] + \frac{1}{2} \sum_{\langle x,y\rangle\in\Lambda} J_{xy}\|\boldsymbol{\phi}(x) - \boldsymbol{\phi}(y)\|^2 - h \sum_{x\in\Lambda} \phi^{(1)}(x)$$

where $\|\boldsymbol{\phi}\|^2 = \sum_{\alpha=1}^{\nu} \phi^{(\alpha)2}$. We suppose that $V(\boldsymbol{\phi})$ depends only on $\|\boldsymbol{\phi}\|$, that there exists $b > 0$ such that $V(\boldsymbol{\phi}) \geqslant a + b\|\boldsymbol{\phi}\|^2$, that $h \geqslant 0$, and that the couplings J_{xy} are nonnegative (ferromagnetic). Although not a necessary restriction, it will simplify the discussion in the following sections to suppose that J_{xy} is translation invariant: $J_{xy} = J(x - y)$ with periodic boundary conditions at the boundary of Λ.

We have then a collection of multicomponent coupled anharmonic oscillators. We will call this model a quantum lattice field, since in the case where the coupling is nearest neighbor and V is appropriately chosen, the limit as the lattice spacing goes to zero will exist and defines a continuum quantum field.[19a,30] The lattice model is called a quantum crystal in Ref. 10.

Following the discussion in Section 2.3, the path space is given by the measure

$$d\mu = \exp\left[-\frac{1}{2} \sum_{\langle x,y\rangle\in\Lambda} J_{xy} \int_0^\beta ds \, \|\hat{\boldsymbol{\phi}}(x, s) - \hat{\boldsymbol{\phi}}(y, s)\|^2\right]$$
$$\times \exp\left\{-\sum_{x\in\Lambda} \int_0^\beta V(\hat{\boldsymbol{\phi}}(x, s))\right\} \prod_{x\in\Lambda} \prod_{\alpha=1}^{\nu} dq_0{}^\alpha(x)\, dw(r^\alpha(x))$$

where dq_0 and dw are as in Section 2.3, and

$$\hat{\phi}^\alpha(x, s) = q_0{}^\alpha(x) + \sqrt{\beta}r^\alpha(x, t/\beta)$$

The lattice approximation allows us to apply correlation inequalities to the expectations of the fields $\hat{\boldsymbol{\phi}}(x, t)$ and then to the Duhamel functions. Assuming the potential $V(\boldsymbol{\phi})$ satisfies the appropriate hypothesis of Section 2.1a–d, we have the following inequalities:

(a) $\nu = 1$ component fields:

$$\begin{aligned}(\phi(y), \phi(y)\phi(z)) &- \langle\phi(y)\phi(z)\rangle\langle\phi(y)\rangle \\ &\leqslant (\phi(y), \phi(y))\langle\phi(z)\rangle + (\phi(y), \phi(z))\langle\phi(y)\rangle \\ &\leqslant \langle\phi(y)^2\rangle\langle\phi(z)\rangle + [\langle\phi(y)^2\rangle\langle\phi(z)^2\rangle]^{1/2}\langle\phi(y)\rangle = 2\langle\phi(y)^2\rangle\langle\phi(y)\rangle\end{aligned}$$

(b) $\nu = 2, 3, 4$ component fields:

$$(\phi^{(2)}(y), \phi^{(1)}(y)\phi^{(2)}(z)) \geqslant 0$$
$$(\phi^{(2)}(y), \phi^{(2)}(y)\phi^{(1)}(z)) \leqslant \langle \phi^{(2)}(y)^2\rangle\langle\phi^{(1)}(z)\rangle$$

These inequalities follow from Eq. (9) for the Duhamel functions in terms of path space expectations, together with the inequalities of Section 2.1, which may be applied via the lattice approximation[19b]; see also Appendix B. We have used the translation invariance of the path space and the fact that $\langle\phi(z)\rangle \geqslant 0$.

2.5. Ising Model with Transverse Field

As another illustration of the relations between classical and quantum statistical mechanics, we consider a simple quantum spin-$\frac{1}{2}$ system.[14,31]

Let $\boldsymbol{\sigma}$ be the Pauli spin matrices, the Hamiltonian $H = -a(\sigma^x - 1)$, $a > 0$. If we compute the kernel of e^{-tH} in a representation which diagonalizes σ^z we obtain

$$\begin{aligned} T(\omega'', \omega') &= \langle \sigma^z = \omega'' | e^{-tH} | \sigma^z = \omega' \rangle \\ &= e^{-ta}\{(\cosh ta)\, \delta_{\omega''=\omega'} + (\sinh ta)\, \delta_{\omega''\neq\omega'}\} \\ &= \tfrac{1}{2}(1 - e^{-4ta})^{1/2} e^{K\omega''\omega'} \end{aligned} \tag{12}$$

where $\omega = \pm 1$, $e^{-2K} = \tanh ta$. That is, $K = (ta)^*$, where the asterisk denotes the usual Ising low-temperature–high-temperature duality transformation.

Since $\sum_{\omega''=\pm1} T(\omega'', \omega') = 1$, we may interpret $T(\omega'', \omega')$ as a probability and construct a path space for the classical random variables $\hat{\sigma}(t) = \pm 1$, $-\beta/2 \leqslant t \leqslant \beta/2$. (We take t between $-\beta/2$ and $+\beta/2$ rather than between 0 and β because the former is appropriate in discussing the limit $\beta \to \infty$.) The path space is determined by the formula

$$\begin{aligned} &\left\langle \prod_{\alpha=1}^{n} F_\alpha(\hat{\sigma}(t_\alpha)) \middle| \text{given } \hat{\sigma}(0) = \omega', \hat{\sigma}(\beta) = \omega'' \right\rangle \\ &= \langle \sigma^z = \omega'' | e^{-(\beta/2 - t_n)H} F_n(\sigma^z) e^{-(t_n - t_{n-1})H} \cdots F_1(\sigma^z) e^{-(t_1+\beta/2)H} | \sigma^z = \omega' \rangle \\ &\quad \times \langle \sigma^z = \omega'' | e^{-\beta H} | \sigma^z = \omega' \rangle^{-1} \end{aligned} \tag{13}$$

Let $d\nu_{\omega''\omega'}$ denote the measure determined by (13). Setting $\omega' = \omega'' = \omega$ and summing over $\omega = \pm 1$ with equal weight gives

$$\left\langle \prod_{\alpha=1}^{n} F_\alpha(\hat{\sigma}(t_\alpha)) \right\rangle_{\text{periodic}} = \frac{\operatorname{Tr}[e^{-(\beta/2 - t_n)H} F_n(\sigma^z) \cdots F_1(\sigma^z) e^{-(t_1+\beta/2)H}]}{\operatorname{Tr} e^{-\beta H}} \tag{14}$$

We denote by $d\nu_{\text{per}}$ the measure corresponding to (14).

Note that the Hamiltonian H is nonnegative and has a unique ground state

$$\Omega = (1/\sqrt{2})(|\sigma^z = 1\rangle + |\sigma^z = -1\rangle), \qquad H\Omega = 0$$

In the limit $\beta \to \infty$, the expectations (13) and (14) converge to the same limit

$$(\Omega, F_n(\sigma^z)e^{-(t_n - t_{n-1})H}F_{n-1}(\sigma^z) \cdots F_1(\sigma^z)\Omega) \tag{15}$$

Introducing a lattice mesh δ by considering only the variables $\hat{\sigma}(n\delta)$, $-\beta/2\delta \leqslant n \leqslant \beta/2\delta$, the path space is seen to be equivalent to a ferromagnetic, nearest neighbor, one-dimensional Ising model with coupling

$$K \sum_n \hat{\sigma}(n\delta)\hat{\sigma}((n+1)\delta), \qquad K = (a\delta)^* = \ln \frac{1}{[\tanh(a\delta)]^{1/2}} \tag{16}$$

as follows from (12).

Let now V be a finite subset of $\mathbb{Z}^s$. A boundary condition is a specification of $\sigma_i^z = \omega_i$ for $i \in V^c$. We then define the Hamiltonian H_V(b.c.), depending on the boundary condition, by

$$\begin{aligned} -H_V(\text{b.c.}) &= a \sum_{i \in V} (\sigma_i^x - 1) + \sum_{\langle i,j \rangle \in V} J_{ij}\sigma_i^z\sigma_j^z + \sum_{i \in V} \sigma_i^z \Big(\sum_{j \in V^c} J_{ij}\omega_j \Big) \\ &= -H_x - H_z \end{aligned}$$

Note that the boundary condition introduces a certain external field in the z direction.

Let $\hat{\boldsymbol{\sigma}}(t)$ denote the set of random variables $\{\hat{\sigma}_i(t)\}_{i \in V}$. With the couplings $J_{ij} = 0$ the path spaces corresponding to (13) and (14) are given by the product measures

$$d\mu^0_{\boldsymbol{\omega}'', \boldsymbol{\omega}'}(\hat{\boldsymbol{\sigma}}) = \bigotimes_{i \in V} d\nu_{\omega_i'', \omega_i'}(\hat{\sigma}_i), \qquad d\mu^0_{\text{per}}(\hat{\sigma}) = \bigotimes_{i \in V} d\nu_{\text{per}}(\hat{\sigma}_i)$$

where $\boldsymbol{\omega}''$, $\boldsymbol{\omega}'$ are specifications of the spins $\hat{\boldsymbol{\sigma}}(\beta)$, $\hat{\boldsymbol{\sigma}}(0)$.

With nonzero couplings

$$\int \prod_{\alpha=1}^{n} F_\alpha(\hat{\boldsymbol{\sigma}}(t_\alpha))\, d\mu_{\boldsymbol{\omega}'', \boldsymbol{\omega}'}(\text{b.c.})$$
$$= \frac{\langle \boldsymbol{\sigma}^z = \boldsymbol{\omega}'' | e^{-(\beta/2 - t_n)H_V(\text{b.c.})} F_n(\boldsymbol{\sigma}^z) \cdots F_1(\boldsymbol{\sigma}^z) e^{-(t_1 + \beta/2)H_V(\text{b.c.})} | \boldsymbol{\sigma}^z = \boldsymbol{\omega}' \rangle}{\langle \boldsymbol{\sigma}^z = \boldsymbol{\omega}'' | e^{-\beta H_V(\text{b.c.})} | \boldsymbol{\sigma}^z = \boldsymbol{\omega}' \rangle} \tag{17}$$

where

$$
\begin{aligned}
&d\mu_{\omega'',\omega'}(\text{b.c.})\\
&= \frac{d\mu^0_{\omega'',\omega'}}{Z_{\omega',\omega''}(\text{b.c.})} \exp\left[\sum_{\langle i,j\rangle \in V} J_{ij} \int_{-\beta/2}^{\beta/2} ds\, \hat{\sigma}_i(s)\hat{\sigma}_j(s) \right.\\
&\quad \left. + \sum_{i\in V} \int_{-\beta/2}^{\beta/2} ds\, \hat{\sigma}_i(s) \Bigl(\sum_{j\in V^c} J_{ij}\omega_j \Bigr) \right]\\
&= \lim_{\delta\to 0} \frac{d\mu^0_{\omega'',\omega'}}{Z^{(\delta)}_{\omega',\omega''}(\text{b.c.})} \exp\left[\sum_{\langle i,j\rangle\in V} \delta J_{ij} \sum_k \hat{\sigma}_i(k\delta)\hat{\sigma}_j(k\delta) \right.\\
&\quad \left. + \sum_{i\in V}\sum_k \hat{\sigma}_i(k\delta) \Bigl(\sum_{j\in V^c} \delta J_{ij}\omega_j \Bigr) \right]
\end{aligned}
\tag{18}
$$

and Z is the normalization integral, so that μ is a probability measure. There are analogous formulas for $d\mu_{\text{per}}(\text{b.c.})$ with a trace in (17). The limit $\delta \to 0$ in (18) equals (17) by the Trotter product formula and the integral over s in (18) is understood as the strong limit of Rieman sums as $\delta \to 0$. The strong continuity of $\hat{\sigma}(t) \to \hat{\sigma}(t + h)$ in L_p, $1 \leqslant p < \infty$, follows from the strong continuity of e^{-tH_x} (see, for example, Ref. 25) or by explicit computation from (12).

Again, as $\beta \to \infty$, the expectations converge to the limit

$$(\Omega_V(\text{b.c.}), F_n(\boldsymbol{\sigma}^z) \exp[-(t_n - t_{n-1})\hat{H}_V(\text{b.c.})] \cdots F_1(\boldsymbol{\sigma}^z)\Omega_V(\text{b.c.})) \tag{19}$$

The path spaces determined by (18) and the analogous formula for the periodic case are in fact continuum Ising models, the limit as $\delta \to 0$ of an Ising model with coupling in the "space" direction δJ_{ij} (small) and coupling in the β direction $K = (a\delta)^*$ (large), as follows from (18) and (16).

The Dirichlet boundary condition for the path space will be useful for the discussion in Section 10.1 and is obtained by removing all couplings to the complement of the region under consideration. To remove the couplings in the space direction we simply set $J_{ij} = 0$ if $i \in V$, $j \in V^c$, or equivalently we set $\omega_j = 0$ in the definition of $H_V(\text{b.c.})$ in order to obtain $H_V{}^D$. To obtain Dirichlet boundary conditions at $t = +\beta/2$ for the path space with mesh δ it is easily seen that we must sum over all possible boundary conditions $\hat{\sigma}(\beta/2)$, with the correct statistical weight:

$$\exp\Bigl[\delta \sum J_{ij}\hat{\sigma}_i{}^z(\beta/2)\hat{\sigma}_j{}^z(\beta/2)\Bigr] = \exp[-\delta H_z(\sigma^z = \hat{\sigma}(\beta/2))]$$

and similarly for $t = -\beta/2$.

Thus the path space with mesh δ and Dirichlet boundary conditions is determined by the formula

$$\left\langle \prod_{\alpha=1}^{n} F_\alpha(\hat{\sigma}(t_\alpha)) \right\rangle_{\beta,V,\delta}^{D}$$

$$= \Big[\sum_{\omega'',\omega'} \langle \boldsymbol{\sigma}^z = \boldsymbol{\omega}'' | e^{-\delta H_z} (e^{-\delta H_x} e^{-\delta H_z})^{(\beta/2 - t_n)/\delta}$$

$$\times F_n(\boldsymbol{\sigma}^z) \cdots F_1(\boldsymbol{\sigma}^z)(e^{-\delta H_x} e^{-\delta H_z})^{(t_1 + \beta/2)/\delta} | \boldsymbol{\sigma}^z = \boldsymbol{\omega}' \rangle \Big]$$

$$\times \Big[\sum_{\omega'',\omega'} \langle \boldsymbol{\sigma}^z = \boldsymbol{\omega}'' | e^{-\delta H_z} (e^{-\delta H_x} e^{-\delta H_z})^{\beta/\delta} | \boldsymbol{\sigma}^z = \boldsymbol{\omega}' \rangle \Big]^{-1}$$

$$\xrightarrow[\delta\to 0]{} \Big\{ \sum_{\omega'',\omega'} \langle \boldsymbol{\sigma}^z = \boldsymbol{\omega}'' | \{\exp[-(\tfrac{1}{2}\beta - t_n) H_V{}^D]\} F_n(\boldsymbol{\sigma}^z) \{\exp[-(t_n - t_{n-1}) H_V{}^D]\}$$

$$\times \cdots F_1(\boldsymbol{\sigma}^z) \exp[-(\tfrac{1}{2}\beta + t_1) H_V{}^D] | \boldsymbol{\sigma}^z = \boldsymbol{\omega}' \rangle \Big\}$$

$$\times \Big[\sum_{\omega'',\omega'} \langle \boldsymbol{\sigma}^z = \boldsymbol{\omega}'' | \exp(-\beta H_V{}^D) | \boldsymbol{\sigma}^z = \boldsymbol{\omega}' \rangle \Big]^{-1}$$

$$\xrightarrow[\beta\to\infty]{} (\Omega_V{}^D, F_n(\boldsymbol{\sigma}^z) \{\exp[-(t_n - t_{n-1}) \hat{H}_V{}^D]\} \cdots F_1(\boldsymbol{\sigma}^z) \Omega_V{}^D) \tag{20}$$

3. UPPER BOUNDS ON THE CRITICAL TEMPERATURE

Let H be the Hamiltonian for a finite-volume lattice model. We write then

$$\langle B \rangle = \frac{\operatorname{Tr} e^{-\beta H} B}{\operatorname{Tr} e^{-\beta H}}$$

for an observable B. In the classical case Tr denotes integration with respect to the product of a priori single site distributions. Let τ_t be a one-parameter group of transformations such that Tr is *invariant* under τ_t, and define then $A' = (d/dt)|_{t=0} \tau_t A$. Then

$$\langle \tau_t A \rangle = \frac{\operatorname{Tr}(e^{-\beta \tau_{-t} H} A)}{\operatorname{Tr} e^{-\beta \tau_{-t} H}} \tag{21}$$

Differentiating (21) at $t = 0$ and using (5), we obtain

$$\langle A' \rangle = \beta (H', A) \tag{22}$$

(since $\langle H' \rangle = 0$ by the invariance of Tr). In the classical case the Duhamel expectation (H', A), (6), is equal to $\langle H'A \rangle$. We will call (22) a *local Ward identity*. The reason for this designation is that in applications there is usually a *global* transformation (e.g., simultaneous rotation of the spins at all lattice points) which leaves the Hamiltonian invariant (and this leads to Ward identities) and there is the *local* transformation (e.g., a rotation of just one spin) which leaves the "free" measure (Tr) invariant but not the Hamiltonian (and this leads to a local form of the Ward identity).

We note that (22) leads, via (7) and (8), to

$$|\langle A'\rangle|^2 \leqslant \beta(H', H'^*)\beta\langle(A^*A + AA^*)/2\rangle = \beta\langle(H'^*)'\rangle\langle(A^*A + AA^*)/2\rangle \tag{23}$$

which is *Bogoliubov's inequality*. {If C is the generator of τ_t, we may express (23) in the more usual form $|\langle[C, A]\rangle|^2 \leqslant \langle[C, [H, C^*]]\rangle\langle(A^*A + AA^*)/2\rangle$}.

The equality (22) leads to a simple estimate of the critical temperature for classical spins and quantum fields.

3.1. Classical One-Component Bounded Spins

We begin with the simple case of one-component bounded spins and derive the mean-field bound, which is slightly weaker than Griffiths' third inequality,[17] but the method may be generalized to the case of quantum unbounded fields. Thus, suppose that σ_i is a real-valued random variable with a priori distribution $d\mu_0(\sigma_i)$ invariant under the transformation $\sigma_i \to -\sigma_i$, and $|\sigma_i| \leqslant b$ with probability one. The Hamiltonian for a finite region has the form

$$-H = \sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j + h\sum \sigma_i \qquad \text{(each pair counted once)}$$

where $h \geqslant 0$ and $J_{ij} \geqslant 0$ (not necessarily nearest neighbor). We note that in general the magnetization is increased by replacing J_{ij} by $|J_{ij}|$ so that the bound will actually apply in all cases. This follows by comparison inequalities (e.g., Refs. 3a and 26).

Replace H by $H(\lambda)$ given by

$$H(\lambda) = \sum_{\substack{\langle i,j\rangle \\ i,j\neq k}} J_{ij}\sigma_i\sigma_j - \lambda\sum_j J_{kj}\sigma_k\sigma_j - h\sum_j \sigma_j$$

When $\lambda = 0$ the spin σ_k is decoupled from the other spins; when $\lambda = 1$ it is fully coupled. When $\lambda = 0$ there is clearly no spontaneous magnetization for the spin σ_k, and we now estimate the effect of restoring the coupling. Letting $\langle\cdots\rangle_\lambda$ denote the expectation with respect to $H(\lambda)$, it follows from (5) and Griffiths' inequalities (Section 2.1a)

$$\begin{aligned}\frac{d}{d\lambda}\langle\sigma_k\rangle_\lambda &= \beta\sum_j J_{kj}[\langle\sigma_k\sigma_j\sigma_k\rangle_\lambda - \langle\sigma_k\sigma_j\rangle_\lambda\langle\sigma_k\rangle_\lambda] \\ &\leqslant \beta\sum_j J_{kj}\langle\sigma_k\sigma_j\sigma_k\rangle_\lambda \leqslant \beta\sum_j J_{kj}\langle\sigma_k{}^2\sigma_j\rangle\end{aligned} \tag{24}$$

Integrating from $\lambda = 0$ to $\lambda = 1$, we find that (24) becomes

$$\langle\sigma_k\rangle \leqslant \langle\sigma_k\rangle_0 + \beta \sum_j J_{kj}\langle\sigma_k{}^2\sigma_j\rangle \tag{25}$$

This formula holds also in the infinite-volume limit.

For the Ising model $\sigma = \pm b$, and we obtain

$$m(h) \leqslant m_0(h) + \beta\mathscr{J}b^2 m(h) \tag{26}$$

where $m(h) = \sup_k\langle\sigma_k\rangle$, $\mathscr{J} = \sup_k \sum_j J_{kj}$, and $m_0(h)$ is the magnetization for a single uncoupled spin.

If $\beta\mathscr{J}b^2 < 1$,

$$m(h) \leqslant \frac{1}{1 - \beta\mathscr{J}b^2}\, m_0(h)$$

Since $m_0(h) \to 0$ as $h \to 0$, it follows that

$$m_+ = 0 \qquad \text{if} \quad \beta\mathscr{J}b^2 < 1 \tag{27}$$

where $m_+ = \lim_{h\downarrow 0} m(h)$.

For the Ising model (27) is slightly weaker than Griffiths' third inequality.[17] For a model with arbitrary even single site distribution and $\sigma_i \in [-b, +b]$ the magnetization is bounded by the magnetization of an Ising model with $\sigma_i = \pm b$ (as follows by a Griffiths' inequality), so that the estimate (27) holds in general. However, we may also note that (26) follows from (25) in general by the following result:

Lemma 3.1. Let $\{\sigma_i\}$ be a set of random variables with even a priori distribution $d\mu_i(\sigma_i)$, $|\sigma_i| \leqslant b$. Let $-\beta H = \sum J_A\sigma_A$, where $J_A \geqslant 0$ and $\sigma_A = \prod_{i\in A}\sigma_i$. Then $\langle\sigma_i{}^2\sigma_B\rangle \leqslant b^2\langle\sigma_B\rangle$.

Proof. Clearly $\int (b^2 - \sigma^2)\sigma^a\, d\mu(\sigma) \geqslant 0$ for any positive integer a. In the usual way, by expanding the exponential $e^{-\beta H}$ and using the factorization of the free a priori measure, the lemma follows from this inequality.

3.2. One-Component Quantum Lattice Fields

We may extend the method of Section 3.1 to quantum lattice fields by using the GHS inequality (Section 2.1b). As in Section 3.1, we obtain from (5) and the GHS inequality (Section 2.4a)

$$\begin{aligned}\frac{d}{d\lambda}\langle\phi(x)\rangle_\lambda &= \beta \sum_y J_{xy}[(\phi(x)\phi(y), \phi(x))_\lambda - \langle\phi(x)\phi(y)\rangle_\lambda\langle\phi(x)\rangle_\lambda] \\ &\leqslant 2\beta\mathscr{J}\langle\phi(x)^2\rangle_\lambda\langle\phi(x)\rangle_\lambda\end{aligned}$$

where we have again used a Griffiths' inequality (for the path space fields $\hat{\phi}$), and $\langle\phi(x)\rangle_0$ is the expectation for a single uncoupled anharmonic oscillator (Schrödinger particle). Again we obtain

$$\langle\phi\rangle_+ \equiv \lim_{h\downarrow 0}\langle\phi\rangle_h = 0 \qquad \text{if} \quad 2\beta\mathscr{J}b^2 < 1 \tag{28}$$

where $b^2 = \lim_{h\downarrow 0}\langle\phi^2\rangle_h \equiv \langle\phi^2\rangle_+$.

We will show in Sections 6 and 7 that $\beta b^2 \to 0$ as $\beta \to 0$ and therefore at high temperature $\langle\phi\rangle_+ = 0$.

3.3. Classical Multicomponent Bounded Spin Models

We use the local Ward identity (22) together with correlation inequalities for the ν-component spins ($\nu = 2, 3, 4$) to obtain the mean-field bound, improving by a factor of two the estimate of Brascamp–Lieb[4] (see also Refs. 22 and 42). For fixed-length spins we obtain almost the mean-field bound for *arbitrary* ν. We begin with the case $\nu = 2, 3, 4$.

Thus we consider $\sigma_i \in \mathbb{R}^\nu$ with a priori distribution $d\mu_0(\|\sigma_i\|)$ depending only on $\|\sigma_i\|$ and $\|\sigma_i\| \leqslant b$ with probability one. The Hamiltonian is as in Section 2.1. We note that for two-component models, a comparison inequality[3a] shows that the magnetization is increased if J_{ij} is replaced by $|J_{ij}|$, so the bound will apply to arbitrary couplings in the two-component case.

Let τ_t denote a rotation by an angle t between the (1) and (2) components of the spin σ_k. Taking $A = \sigma_k^{(2)}$ in Eq. (22), the following equality is obtained:

$$\langle\sigma_k^{(1)}\rangle = \beta \sum_j J_{kj}\langle\sigma_k^{(2)}[\sigma_k^{(2)}\sigma_j^{(1)} - \sigma_k^{(1)}\sigma_j^{(2)}]\rangle + \beta h\langle(\sigma_k^{(2)})^2\rangle$$

Then, supposing the single site distribution satisfies the hypotheses in Section 2.1c, d,

$$\begin{aligned}\langle\sigma_k^{(1)}\rangle &\leqslant \beta \sum_j J_{kj}\langle\sigma_k^{(2)}\sigma_k^{(2)}\sigma_j^{(1)}\rangle + \beta h\langle(\sigma_k^{(2)})^2\rangle \\ &\leqslant \beta \sum_j J_{kj}\langle(\sigma_k^{(2)})^2\rangle\langle\sigma_j^{(1)}\rangle + \beta h\langle(\sigma_k^{(2)})^2\rangle \end{aligned} \tag{29}$$

By symmetry, $\langle(\sigma_k^{(\alpha)})^2\rangle = \langle(\sigma_k^{(2)})^2\rangle$ if $\alpha = 3,\dots,\nu$. Thus again by inequalities $\langle(\sigma_k^{(2)})^2\rangle \leqslant \langle(\sigma_k^{(1)})^2\rangle$, giving

$$\begin{aligned}\nu\langle(\sigma_k^{(\alpha)})^2\rangle &= (\nu - 1)\langle(\sigma_k^{(2)})^2\rangle + \langle(\sigma_k^{(2)})^2\rangle \leqslant (\nu - 1)\langle(\sigma_k^{(2)})^2\rangle + \langle(\sigma_k^{(1)})^2\rangle \\ &= \langle\|\sigma_k\|^2\rangle \leqslant b^2\end{aligned}$$

With $m(h) = \sup_k\langle\sigma_k^{(1)}\rangle$, $\mathscr{J} = \sup_k \sum_j J_{kj}$, we obtain from (29)

$$m(h) \leqslant (\beta\mathscr{J}b^2/\nu)m(h) + \beta h(b^2/\nu) \tag{30}$$

which implies

$$m(h) \leqslant \frac{1}{1-(\beta\mathscr{J}b^2/\nu)}\frac{\beta b^2}{\nu}h$$

and so

$$m_+ \equiv \lim_{h\downarrow 0} m(h) = 0 \qquad \text{if} \quad \beta\mathscr{J}b^2/\nu < 1, \quad \nu = 2, 3, 4 \tag{31}$$

Without any assumptions concerning correlation inequalities we may obtain an estimate valid for all ν if we consider *fixed-length* spins, $\|\sigma_i\| = b$. From the local Ward identities we obtain, using Griffiths' inequality 2.1a (which is true for all ν)

$$\langle\sigma_k^{(1)}\rangle \leqslant \beta\sum_j J_{kj}\langle(\sigma_k^{(2)})^2\sigma_j^{(1)}\rangle + \beta h\langle(\sigma_k^{(2)})^2\rangle \tag{32}$$

By symmetry

$$(\nu-1)\langle\sigma_k^{(1)}\rangle \leqslant \beta\sum_j J_{kj}\left\langle\sum_{\alpha=2}^{\nu}(\sigma_k^{(\alpha)})^2\sigma_j^{(1)}\right\rangle + \beta h\left\langle\sum_{\alpha=2}^{\nu}(\sigma_k^{(\alpha)})^2\right\rangle$$

Since $(\sigma_k^{(1)})^2 + \sum_2^{\nu}(\sigma_k^{(\alpha)})^2 = b^2$ we obtain

$$\begin{aligned}(\nu-1)\langle\sigma_k^{(1)}\rangle &\leqslant \beta\sum_j J_{kj}\{b^2\langle\sigma_j^{(1)}\rangle - \langle(\sigma_k^{(1)})^2\sigma_j^{(1)}\rangle\} + \beta h\{b^2 - \langle(\sigma_k^{(1)})^2\rangle\}\\ &\leqslant \beta\sum_j J_{kj}b^2\langle\sigma_j^{(1)}\rangle + \beta b^2 h\end{aligned}$$

This leads to

$$(\nu-1)m(h) \leqslant \beta\mathscr{J}b^2m(h) + \beta hb^2$$

Thus

$$m_+ = \lim_{h\downarrow 0} m(h) = 0 \qquad \text{if} \quad [\beta Jb^2/(\nu-1)] < 1 \tag{33}$$

Note that (33) differs from (31) by the replacement $\nu \to \nu - 1$.

3.4. Multicomponent Quantum Lattice Fields

For certain local potentials $V(\|\phi\|^2)$ absence of spontaneous field expectation follows from the same result for one-component fields using correlation inequalities.[2] More generally we may proceed by applying inequalities to the Duhamel function by the method of Section 3.3.

As in Section 3.3 we obtain from (22) for $\nu = 2, 3, 4$

$$\begin{aligned}\langle\phi^{(1)}(x)\rangle &= \beta\sum_y J_{xy}(\phi^{(2)}(x), [\phi^{(2)}(x)\phi^{(1)}(y) - \phi^{(1)}(x)\phi^{(2)}(y)])\\ &\quad + \beta h(\phi^{(2)}(x), \phi^{(2)}(x))\\ &\leqslant \beta\sum_y J_{xy}\langle\phi^{(2)}(x)^2\rangle\langle\phi^{(1)}(y)\rangle + \beta h\langle\phi^{(2)}(x)^2\rangle\end{aligned}$$

where we have used the inequalities of Section 2.4.

Since $\beta\langle\phi^{(2)}(x)^2\rangle \to 0$ as $\beta \to 0$ by the results of Sections 6 and 7 we may again show the absence of spontaneous field expectation at high enough temperatures as in Section 3.2.

3.5. Upper Bound on the Critical Temperature for Bounded Classical Spins

We summarize here the results of Sections 3.1 and 3.3 for classical bounded spins.

Theorem 3.1. Let $\sigma_i \in \mathbb{R}^\nu$ be ν-component classical spins, $\|\sigma_i\| \leqslant b$. The single site distribution $d\mu_0(\|\sigma_i\|)$ satisfies:

(a) $\nu = 1$; $d\mu_0$ is arbitrary (even).

(b) $\nu = 2$; $d\mu_0$ is a limit of measures of the form $\exp[-V(\|\sigma\|^2)]\, d\sigma^{(1)}\, d\sigma^{(2)}$, $V(\xi)$ convex.

(c) $\nu = 3, 4$; $d\mu_0$ is a limit of measures of the form $[\exp(-a\|\sigma\|^4 + b\|\sigma\|^2)] \times \prod_{i=1}^{\nu} d\sigma^{(i)}$.

(d) $\nu \geqslant 5$; $d\mu_0$ is fixed length, $\delta(\|\sigma\|^2 - b^2)\prod_{i=1}^{\nu} d\sigma^{(i)}$.

The Hamiltonian has the form $-H = \sum_{\text{pairs}} J_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i$, $J_{ij} \geqslant 0$, $h \geqslant 0$ (we suppose $J_{ii} = 0$ for all i). Let $\mathcal{J} = \sup_i \sum_j J_{ij}$. *Then* the spontaneous magnetization is zero if

$$\beta < \nu/\mathcal{J} b^2\alpha$$

where $\alpha = 1$ if $\nu = 1, 2, 3, 4$ and $\alpha = \nu/(\nu - 1)$ for $\nu \geqslant 5$.

If we combine Theorem 3.1 with FKG inequalities for $\nu = 1$ and generalized Griffiths' inequalities for $\nu = 2$ as in Refs. 3a and 27, we obtain the following result:

Corollary 3.1. With the hypothesis of Theorem 3.1, if $\beta < \nu/\mathcal{J} b^2$, *then*:

(a) $\nu = 1$: There is a unique equilibrium state.

(b) $\nu = 2$: There is a unique phase.

4. EXISTENCE OF CRITICAL LENGTH

Here we will use the estimates of Section 3 together with the method of infrared bounds[11] to prove the existence of a critical length for certain (multicomponent) classical systems. Consider a one-dimensional ferromagnetic Ising model with nearest neighbor interactions on the lattice $\mathbb{Z}$. There is no spontaneous magnetization and indeed a unique equilibrium state by the transfer matrix formalism. Consider now a two-dimensional Ising model, infinite in one dimension and finite in the other with length[4] L and periodic boundary conditions. We denote this lattice by $\mathbb{Z} \times L$. For any given L the system is effectively one-dimensional and again by the transfer matrix there is no spontaneous magnetization. However, for $L = \infty$ the lattice is $\mathbb{Z}^2$ and for sufficiently large β the spontaneous magnetization is nonzero. Thus in this case there is no critical length for spontaneous magnetization. If we now consider an infinite, two-dimensional lattice with a finite periodic third dimension so that the lattice is $\mathbb{Z}^2 \times L$, then there is a long-range order for β large enough if $L = 1$ (the system is then strictly two-dimensional) and therefore also for any L, $1 \leqslant L \leqslant \infty$, by Griffiths' inequalities. One would expect that if β is chosen so that $\beta_c(3) < \beta < \beta_c(2)$,[5] there would be a critical length. (Actually we do not consider this case; our discussion applies to $\mathbb{Z}^s \times L$, $s \geqslant 3$.)

Consider now a one-dimensional, nearest neighbor, ferromagnetic, fixed-length rotator model on the lattice $\mathbb{Z}$. Again by the transfer matrix formalism there is no spontaneous magnetization and the same is true for the lattice $\mathbb{Z} \times L$. In fact on the two-dimensional lattice $\mathbb{Z}^2$ there is still no spontaneous magnetization by Mermin's theorem[(28)] and the theorem applies also to the lattice $\mathbb{Z}^2 \times L$. However, for the three-dimensional lattice $\mathbb{Z}^3$ there is spontaneous magnetization.[(11)] Here also, then, there is no critical length. However, for the lattice $\mathbb{Z}^3 \times L$, one would expect that if $\beta_c(4) < \beta < \beta_c(3)$ there would be a critical length.

Actually with periodic boundary conditions we cannot prove monotonicity of the spontaneous magnetization in L, and so we cannot show that if the spontaneous magnetization is nonzero for $L = L_1$ it is nonzero for all $L > L_1$. Instead we show there are lengths $L_1 \leqslant L_2$ such that for all $L \leqslant L_1$ there is no spontaneous magnetization and for all $L > L_2$ there is spontaneous magnetization (presumably $L_1 = L_2$). For nearest neighbor interactions we only know that $L_1 \geqslant 1$.

We consider then the lattice $\mathbb{Z}^s \times L$, $s \geqslant 3$, with nearest neighbor ferromagnetic pair interactions and fixed-length spins ($\|\sigma\| = b$). The Hamiltonian is $H = -J \sum_{\langle x,y\rangle} \sigma_x \cdot \sigma_y$ or equivalently $H = (J/2) \sum_{\langle x,y\rangle} \|\sigma_x - \sigma_y\|^2$, where $\langle x, y\rangle$ runs through all nearest neighbor pairs, each pair counted once. The periodic rectangular lattice Λ is

$$\Lambda = V \times L = \{x = (x_0, x_1,..., x_s) = (x_0, \mathbf{x})\colon x_\alpha = n_\alpha, n_\alpha = 0, 1,..., L_\alpha - 1\}$$

with L_α the length in direction α and $L = L_0$. The dual lattice Λ^* is

$$\Lambda^* = \{p = (p_0, p_1,..., p_s\} = (p_0, \mathbf{p})\colon p_\alpha = (2\pi/L_\alpha)n_\alpha, n_\alpha = 0, 1,..., L_\alpha - 1\}$$

[4] L is the number of copies of $\mathbb{Z}$, and is an integer.

[5] $\beta_c(s)$ denotes the critical temperature for the lattice $\mathbb{Z}^s$.

Let $\Delta p_\alpha = 2\pi/L_\alpha$, $\Delta p = \prod_{\alpha=0}^{s} \Delta p_\alpha = (2\pi)^d/|\Lambda|$, $\Delta \mathbf{p} = \prod_{\alpha=1}^{s} \Delta p_\alpha$, $d = s + 1$, $|\Lambda| = \prod_{\alpha=0}^{s} L_\alpha$. The infrared bound of FSS[11] is

$$\langle \sigma_{\mathbf{x}}^{(\alpha)} \sigma_{\mathbf{y}}^{(\alpha)} \rangle = \frac{1}{2\pi} \sum_{p_0} \Delta p_0 \frac{1}{(2\pi)^s} \sum_{\mathbf{p}} \Delta \mathbf{p} \{\exp[i\mathbf{p}\cdot(\mathbf{y} - \mathbf{x})]\} \rho_\Lambda(p_0, \mathbf{p}) \tag{34}$$

the function $\rho_\Lambda(p_0, \mathbf{p})$ satisfying the bound

$$0 \leqslant \rho_\Lambda(p_0, \mathbf{p}) \leqslant \frac{1}{2\beta J} \frac{1}{(1 - \cos p_0) + \sum_{\alpha=1}^{s} (1 - \cos p_\alpha)} \tag{35}$$

We now take the limit $V \nearrow \mathbb{Z}^s$. Away from $(p_0, \mathbf{p}) = (0, \mathbf{0})$ the sum in (34) converges to the Riemann integral and, *assuming no long-range order* [no $\delta(\mathbf{p})\, \delta p_0$ contribution], Eqs. (34) and (35) imply

$$\langle (\sigma_{\mathbf{x}}^{(\alpha)})^2 \rangle \leqslant \frac{1}{2\beta J} \frac{1}{2\pi} \sum_{p_0} \Delta p_0 \frac{1}{(2\pi)^s} \int_{0 \leqslant p_\alpha \leqslant 2\pi} d^s\mathbf{p} \frac{1}{(1 - \cos p_0) + \sum_{\alpha=1}^{s} (1 - \cos p_\alpha)}$$

Summing over α, we obtain

$$\begin{aligned} b^2 &= \langle \|\sigma_{\mathbf{x}}\|^2 \rangle \\ &\leqslant \frac{\nu}{2\beta J} \frac{1}{2\pi} \sum_{p_0} \Delta p_0 \frac{1}{(2\pi)^s} \int_{0 \leqslant p_\alpha \leqslant 2\pi} d^s\mathbf{p} \frac{1}{(1 - \cos p_0) + \sum_{\alpha=1}^{s} (1 - \cos p_\alpha)} \end{aligned} \tag{36}$$

(Note that for $s = 1, 2$, the integral with $p_0 = 0$ diverges). For $s \geqslant 3$ the integral in (36) is a continuous function of p_0 and it follows that for $L \to \infty$ the right-hand side of (36) converges to $(\nu/2\beta J) I_d$, where

$$I_d = (2\pi)^{-d} \int_{0 \leqslant p_\alpha \leqslant 2\pi} d^d p \frac{1}{\sum_{\alpha=1}^{d} (1 - \cos p_\alpha)}$$

If β is large enough so that

$$(\nu/2\beta J) I_d < b^2 \tag{37}$$

we obtain a contradiction with (36) for L large enough. Thus (37) implies the existence of L_c such that for $L > L_c$ there is long-range order (and a nonzero spontaneous magnetization). To establish the existence of a critical length we must show that at least for $L = 1$ there is no spontaneous magnetization. This is accomplished using the bounds of Section 3. From these it follows that if

$$(\nu/2Jb^2) I_d < \beta < \nu/\mathscr{J} b^2 \alpha \tag{38}$$

then for that value of β there exists a critical length. For an s-dimensional lattice with nearest neighbor interactions $\mathscr{J} = 2sJ$ since there are $2s$ nearest neighbors. We must thus establish the bound $I_d < (d - 1)^{-1} \alpha^{-1}$. In Section 5 we prove $I_d < (d - 1)^{-1}$ for all $d \geqslant 4$; thus we have the following result:

Theorem 4.1. There exists a critical length for nearest neighbor, ferromagnetic, pair-interacting, fixed-length spins on the lattice $\mathbb{Z}^s \times L$ for $s \geqslant 3$ if the number of components $\nu = 1, 2, 3, 4$ or if ν is sufficiently large [so that $\nu(\nu - 1)^{-1}$ is close enough to 1].

Note that Eq. (37) shows[11]

$$T_c \geqslant \frac{2Jb^2}{k}\frac{1}{\nu I_d} \tag{39}$$

where T_c is the critical temperature for spontaneous magnetization on the lattice $\mathbb{Z}^d$.

5. ON ESTIMATING LATTICE GREEN'S FUNCTIONS

We develop a procedure which is useful for getting sharp estimates for the numbers ($d \geqslant 3$)

$$I_d = \frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi} d\phi_1 \cdots d\phi_d \left[\sum_1^d (1 - \cos\phi_i)\right]^{-1} \tag{40}$$

I_3 has been calculated exactly in Ref. 40, but for I_d, $d \geqslant 4$, little seems to be known beyond DLS.[6] A direct numerical computation of I_d is hampered by the fact of numerical instability and even for I_4 a computer calculation is difficult.[6]

We start by noting that for $d \geqslant 3$

$$I_d = \int_0^\infty e^{-dt} f(t)^d \, dt \tag{41}$$

where

$$f(t) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, e^{t\cos\phi} = \sum_0^\infty \frac{(t/2)^{2n}}{(n!)^2} = \sum_0^\infty \frac{(t/2)^{2n}}{(2n)!}\binom{2n}{n} \tag{42}$$

From this we get

$$f(t)^2 = \sum_0^\infty \left(\frac{t}{2}\right)^{2n} \frac{1}{(n!)^2} \sum_{l=0}^{n} \binom{n}{l}^2$$

which by Vandermonde's identity becomes

$$f(t)^2 = \sum_0^\infty \left(\frac{t}{2}\right)^{2n} \frac{1}{(n!)^2}\binom{2n}{n} \tag{43}$$

We rewrite $f(t)^2$ as

$$f(t)^2 = \sum_0^\infty \frac{(2t)^{2n}}{(2n+1)!} C_n \tag{44}$$

where $C_n = (2n+1)!(2n)!/(2^n n!)^4$, $C_0 = 1$, $C_1 = 3/4$, $C_2 = 45/64$,.... Clearly $C_{n+1} < C_n \;\forall n \in \mathbb{N}$ and by applying Stirling's formula one easily finds $\lim_{n\to\infty} C_n = 2/\pi$.

Definition. An nth-order approximation to $f(t)^2$ is a power series in which all C_k, $k \geqslant n$, are replaced by C_n (upper bound) or all C_k, $k \geqslant n$, are replaced by $2/\pi$ (lower bound).

[6] The authors are indebted to F. Driessler for a computer study of I_4.

So, in first order we have

$$\left(1-\frac{2}{\pi}\right)+\frac{2}{\pi}\frac{\sinh 2t}{2t}<f(t)^2<\frac{1}{4}+\frac{3}{4}\frac{\sinh 2t}{2t}$$

and in second order

$$\left(1-\frac{2}{\pi}\right)+\left(\frac{3}{4}-\frac{2}{\pi}\right)\frac{2}{3}t^2+\frac{2}{\pi}\frac{\sinh 2t}{2t}<f(t)^2<\frac{19}{64}+\frac{1}{32}t^2+\frac{45}{64}\frac{\sinh 2t}{2t}$$

Since all integrals arising from a substitution for f^2 in (41) for even d can be carried out explicitly, we have here a method for actually calculating I_d, d even. For I_4 we then get in first order $0.3005 < I_4 < 0.316$ ($<1/3$) and in second order $0.3069 < I_4 < 0.311$. In first order we obtain $I_6 < 0.18652$ ($<1/5$). Writing $I_5 \leqslant (I_4 I_6)^{1/2}$, we immediately get $I_5 \leqslant 0.241$ ($<1/4$).

As regards a global bound, we now prove the following result:

Theorem 5.1. For $d \geqslant 4$, $(d-1/2)^{-1} < I_d < [d-\alpha(d)]^{-1} < (d-1)^{-1}$, where $\alpha(d) \to 1/2$ for $d \to \infty$.

Proof. (a) Lower bound: From (41) we obtain[7]

$$dI_d = \int_0^\infty dx\, e^{-x} f(x/d)^d \tag{45}$$

Integrating by parts twice, we have

$$\begin{aligned} dI_d &= 1 + \int_0^\infty dx\, e^{-x} f\left(\frac{x}{d}\right)^{d-1} f'\left(\frac{x}{d}\right) \qquad (46)\\ &= 1 + \int_0^\infty dx\, e^{-x}\left\{\frac{d-1}{d} f\left(\frac{x}{d}\right)^{d-2} f'\left(\frac{x}{d}\right)^2 + \frac{1}{d} f\left(\frac{x}{d}\right)^{d-1} f''\left(\frac{x}{d}\right)\right\}\\ &= 1 + \frac{1}{d} f''(0)\int_0^\infty dx\, e^{-x} f\left(\frac{x}{d}\right)^{d-1}\\ &\quad + \frac{1}{d}\int_0^\infty dx\, x^2 e^{-x}\left\{\frac{d-1}{d^2} f\left(\frac{x}{d}\right)^{d-2}\left[\frac{f'(x/d)}{(x/d)}\right]^2\right.\\ &\quad \left. + \frac{1}{d^2} f\left(\frac{x}{d}\right)^{d-1}\left[\frac{f''(x/d)-f''(0)}{(x/d)^2}\right]\right\}\\ &\geqslant 1 + \frac{1}{2d} + \frac{1}{d}\int_0^\infty dx\, x^2 e^{-x}\left\{\frac{d-1}{d^2}\left[\frac{1}{2}\right]^2 + \frac{1}{d^2}\frac{3}{16}\right\}\\ &= 1 + \frac{1}{2d} + \frac{1}{2}\frac{d-1}{d^3} + \frac{3}{8}\frac{1}{d^3} > \frac{d}{d-\frac{1}{2}} \qquad (47) \end{aligned}$$

[7] We remark that $f(x/d)^{d/x} = [(1/2\pi)\int_0^{2\pi} d\theta\, e^{(d/x)\cos\theta}]^{d/x} = \|e^{\cos\theta}\|_{x/d}$ is an L_p norm with respect to the probability measure $d\theta/2\pi$. It follows that $f(x/d)^d$ monotone decreases as d increases, converging to 1 [since $f'(0) = 0$]. Therefore dI_d decreases to 1 as $d \nearrow \infty$. This is a result of Ref. 6.

In obtaining (47) we have used $f(t) \geqslant 1$, $t^{-1}f'(t) \geqslant 1/2$, $t^{-2}[f''(t) - f''(0)] \geqslant 3/16$, and $f''(0) = 1/2$, all of which follow from (42) since all coefficients in the power series for f are positive.

(b) Upper bound: It follows from (42) that $t^{-1}f'(t) \leqslant (1/2)f(t)$. Substituting this into (46) gives

$$dI_d \leqslant 1 + \frac{1}{2d}\int_0^\infty dx\, xe^{-x} f\left(\frac{x}{d}\right)^d \tag{48}$$

Since $f(x/d)^d$ is monotone decreasing as d increases (see footnote 7), we have

$$dI_d \leqslant 1 + \frac{1}{2d}\int_0^\infty dx\, xe^{-x} f\left(\frac{x}{d_0}\right)^{d_0} \tag{49}$$

for any $d_0 \leqslant d$. We have already shown $I_d < (d-1)^{-1}$ for $d = 4, 5, 6$. For $d > 6$ we take $d_0 = 6$ and obtain

$$dI_d \leqslant 1 + \frac{1}{2d}\int_0^\infty dx\, xe^{-x} f\left(\frac{x}{6}\right)^6 < 1 + \frac{1}{2d}\frac{25}{16} < \frac{d}{d-1}$$

where we have substituted the first-order upper bound for $f(t)^2$.

(c) Finally we note that

$$\int_0^\infty dx\, xe^{-x} f\left(\frac{x}{d}\right)^d \searrow \int_0^\infty dx\, xe^{-x} = 1 \qquad \text{as } d \nearrow \infty$$

So for $d \geqslant 6$ if $\alpha(d) = \frac{1}{2}\int_0^\infty dx\, xe^{-x} f(x/d)^d$ we have from (48)

$$I_d \leqslant \frac{1}{d}\left(1 + \frac{\alpha(d)}{d}\right) < \frac{1}{d - \alpha(d)} \qquad \text{and} \qquad \alpha(d) \to \frac{1}{2} \qquad \text{for } d \to \infty$$

The upper bound for I_d given in Theorem 5.1, together with Theorem 3.1 and Eq. (39), gives the following bounds for the critical temperature of classical fixed-length spins interacting via nearest neighbor ferromagnetic pair interactions on the lattice $\mathbb{Z}^d$.

Theorem 5.2. Let $\sigma_i \in \mathbb{R}^\nu$ be ν-component classical spins of fixed length $\|\sigma_i\| = b$ on the lattice $\mathbb{Z}^d$ with Hamiltonian $-H = J\sum_{\langle i,j\rangle} \sigma_i \cdot \sigma_j + h\sum \sigma_i^{(1)}$ with $J \geqslant 0$ and $\langle i,j\rangle$ running over all nearest neighbor pairs. The critical temperature T_c for spontaneous magnetization satisfies

$$2\frac{Jb^2}{k}\frac{d-1}{\nu} < T_c \leqslant 2\frac{Jb^2}{k}\frac{d}{\nu}\alpha \qquad (d \geqslant 4)$$

where k is Boltzmann's constant and $\alpha = 1$ if $\nu = 1, 2, 3, 4$; $\alpha = \nu(\nu-1)^{-1}$ if $\nu \geqslant 5$. For $d = 4$, $d - 1$ in the lhs may be replaced by 3.2144. For $d = 3$, $d - 1$ in the lhs must be replaced by 1.978.[11,40]

In particular for $\nu = 1, 2, 3, 4$, $T_c/d \to_{d\to\infty} T_c(\text{mean field})/d = 2Jb^2/k\nu$, extending to $\nu = 2, 3, 4$ the observation of Ref. 6 for $\nu = 1$.

6. ABSENCE OF SYMMETRY BREAKING FOR QUANTUM LATTICE FIELDS AT HIGH TEMPERATURE

Spontaneous symmetry breaking does occur at sufficiently low temperature for quantum lattice fields on the lattice $\mathbb{Z}^s$, $s \geqslant 3$, by infrared bounds,[10,11] and for one-component fields for $s = 2$, by the Peierls argument.[10,18] We review these arguments in Section 9. Using correlation inequalities (Section 2) and local Ward identities (Section 3), we will show that spontaneous magnetization (field expectation) does not occur at sufficiently high temperature (β small enough). Indeed, as pointed out in Sections 3.2 and 3.4, the estimate $\beta\langle\phi(x)^2\rangle \xrightarrow[\beta\to 0]{} 0$ suffices.

The high-temperature behavior of quantum fields (or even a Schrödinger particle) is somewhat subtle since the kinetic energy diverges, as do the correlation functions (even for the free field). Renormalizations have to be done (depending on the interaction) in order to have uniformly bounded correlation functions for all β. This is in contrast to the usual lattice spin systems where the high-temperature limit is trivial (there is no kinetic energy in this case).

Consider a one-component lattice field $\phi(x)$ with local potential

$$V(\phi) = \sum_{k=1}^{n} a_k\phi^{2k} - h\phi, \qquad n \geqslant 2, h \geqslant 0, a_n > 0, a_k \geqslant 0 \quad \text{for } k \geqslant 2 \tag{50}$$

on a finite rectangular sublattice Λ of $\mathbb{Z}^s$ with periodic boundary conditions and translation-invariant pair interaction $J(x, y) = J(x - y)$:

$$H = \sum_{x\in\Lambda} \tfrac{1}{2}\pi(x)^2 + V(\phi(x)) + \tfrac{1}{2} \sum_{\text{pairs}} J_{xy}[\phi(x) - \phi(y)]^2 \tag{51}$$

Expectations are given by $\langle A\rangle = \mathrm{Tr}(e^{-\beta H}A)/\mathrm{Tr}(e^{-\beta H})$. The Peierls-Bogoliubov inequality[33] gives,[8]

$$\exp[\beta\epsilon\langle\phi(x)^{2n}\rangle] = \exp\left[\left\langle \beta\epsilon \sum_{x\in\Lambda} \phi(x)^{2n}\right\rangle \frac{1}{|\Lambda|}\right] \leqslant \left[\frac{\mathrm{Tr}\exp(-\beta\tilde{H})}{\mathrm{Tr}\exp(-\beta H)}\right]^{1/|\Lambda|} \tag{52}$$

where $\tilde{H}$ has the form (51) with V as in Eq. (50) but with a_n replaced by $a_n - \epsilon$. We estimate the right-hand side of (52) in terms of single uncoupled fields by removing the couplings by means of the operator inequalities

$$0 \leqslant [\phi(x) - \phi(y)]^2 \leqslant 2[\phi(x)^2 + \phi(y)^2]$$

[8] If formulated in terms of path space expectations eq. (6.3) easily follows from Jensen's inequality.

which imply

$$\tilde{H} \geqslant \sum_{x \in \Lambda} \tilde{H}_{(1)}(x), \qquad H \leqslant \sum_{x \in \Lambda} H_{(1)}(x)$$

where

$$\begin{aligned}
\tilde{H}_{(1)}(x) &= \tfrac{1}{2}\pi(x)^2 + \tilde{V}(\phi(x)) \\
H_{(1)}(x) &= \tfrac{1}{2}\pi(x)^2 + V(\phi(x)) + \mathscr{J}\phi(x)^2 = \tfrac{1}{2}\pi(x)^2 + \hat{V}(\phi(x)) \\
\mathscr{J} &= \sum_y J_{xy}
\end{aligned}$$

Then Eq. (52) gives

$$\exp[\beta\epsilon\langle\phi(x)^{2n}\rangle] \leqslant \frac{\operatorname{Tr}\exp(-\beta\tilde{H}_{(1)})}{\operatorname{Tr}\exp(-\beta H_{(1)})} \tag{53}$$

We conclude that $\beta\langle\phi(x)^{2n}\rangle$ is uniformly bounded as $\beta \to 0$ (and uniformly in the volume Λ) if the right-hand side of (53) is uniformly bounded. Then using $\langle\phi(x)^2\rangle^n \leqslant \langle\phi(x)^{2n}\rangle$ (by a Griffiths inequality) we have $\beta^{1/n}\langle\phi(x)^2\rangle \leqslant C$ as $\beta \to 0$ and thus $\beta\langle\phi(x)^2\rangle \to 0$, which is the desired result. (Clearly the same discussion applies to multicomponent fields.)

To bound the right-hand side of (53) we show in Section 7 that its limit as $\beta \to 0$ is classical:

$$\lim_{\beta\to 0}\frac{\operatorname{Tr}\exp(-\beta\tilde{H}_{(1)})}{\operatorname{Tr}\exp(-\beta H_{(1)})} = \lim_{\beta\to 0}\frac{\int dx\,\exp[-\beta\tilde{V}(x)]}{\int dx\,\exp[-\beta\hat{V}(x)]} = \left(\frac{a_n}{a_n - \epsilon}\right)^{1/(2n)}$$

7. THE HIGH-TEMPERATURE LIMIT IS CLASSICAL

For finitely many degrees of freedom we show that $\operatorname{Tr}\exp\{-\beta[\frac{1}{2}\pi^2 + V(q)]\}$ approaches its classical value as $\beta \to 0$.[9]

Using the representation (10) and making the change of variables $y = \beta^{1/(2n)}x$, we have

$$\begin{aligned}
&\operatorname{Tr}\exp\{-\beta[\tfrac{1}{2}\pi^2 + V(q)]\} \\
&\quad = (2\pi\beta)^{-1/2}\beta^{-1/(2n)}\int_{-\infty}^{\infty} dy \\
&\left\langle \exp\left\{-\int_0^1 dt\left[\sum_{k=1}^{n} a_k\beta^{1-k/n}(y + \beta^{1/2+1/(2n)}r(t))^{2k}\right.\right.\right. \\
&\qquad \left.\left.\left. - h\beta^{1-1/(2n)}(y + \beta^{1/2+1/(2n)}r(t))\right]\right\}\right\rangle_r
\end{aligned}$$

[9] The limit $\beta \to 0$ should not be confused with the related $\hbar \to 0$ limit.

Since $r(t)$ is continuous along each path (Appendix B), the integral in the exponential converges to $a_n y^{2n}$ as $\beta \to 0$ for each path. Then

$$\left\langle \exp\left\{ -\int_0^1 dt\, [\cdots] \right\} \right\rangle_r \xrightarrow[\beta\to 0]{} \exp(-a_n y^{2n})$$

since $\exp\{-\int_0^1 dt\, [\cdots]\}$ is bounded uniformly in β. To obtain the limit $\beta \to 0$ for the integral $\int_{-\infty}^{\infty} dy\, \langle \cdots \rangle_r$ we use Lebesgue-dominated convergence. By Jensen's inequality

$$\begin{aligned} &\left\langle \exp\left\{ -\int_0^1 dt\, [\cdots] \right\} \right\rangle_r \\ &\quad \leqslant \int_0^1 dt\, \langle \exp\{-[\cdots]\} \rangle_r \\ &\quad = \left\langle \exp\left\{ -\left[\sum_{k=1}^{n} a_k \beta^{1-k/n} (y + \beta^{1/2+1/(2n)} r(0))^{2k} \right.\right.\right. \\ &\qquad \left.\left.\left. - h\beta^{1-1/(2n)} (y + \beta^{1/2+1/(2n)} r(0)) \right] \right\} \right\rangle_r \end{aligned}$$

the last equality following from the translation invariance of the path space. Then

$$\left\langle \exp\left\{ -\int_0^1 dt\, [\cdots] \right\} \right\rangle_r \leqslant C \langle \exp\{-a_n'(y + \beta^{1/2+1/(2n)} r(0))^{2n}\} \rangle_r$$

for some $a_n' < a_n$ (a_n' is independent of β). Thus

$$\begin{aligned} &\left\langle \exp\left(-\int_0^1 dt\, [\cdots] \right) \right\rangle \\ &\qquad \leqslant C' \int_{-\infty}^{\infty} dr\, [\exp(-\tfrac{1}{2} a r^2)] \exp[-a_n'(y + \beta^{1/2+1/(2n)} r)^{2n}] \end{aligned}$$

For $\beta \leqslant 1$,

$$\begin{aligned} &\exp\{-\tfrac{1}{2} a r^2 - a_n'(y + \beta^{1/2+1/(2n)} r)^{2n}\} \\ &\quad \leqslant \chi(\operatorname{sign} y = \operatorname{sign} r) \exp(-\tfrac{1}{2} a r^2 - a_n' y^{2n}) \\ &\qquad + \chi(\operatorname{sign} y = -\operatorname{sign} r) \chi(|y| \leqslant 2|r|) \exp(-\tfrac{1}{2} a r^2) \\ &\qquad + \chi(\operatorname{sign} y = -\operatorname{sign} r) \chi(|y| \geqslant 2r) \exp\{-[\tfrac{1}{2} a r^2 + a_n' y^{2n}/2^{2n}]\} \end{aligned} \tag{54}$$

where $\chi(P)$ denotes the characteristic function of the set determined by the property P. Clearly the integral over r of the right-hand side of (54) (which is independent of β) is an element of $L_1(dy)$. We have thus proved the following result:

Theorem 7.1.

$$\lim_{\beta\to 0} \beta^{1/2+1/(2n)} \operatorname{Tr} \exp\{-\beta[\tfrac{1}{2}\pi^2 + V(q)]\} = (2\pi)^{-1/2} \int_{-\infty}^{\infty} dy \exp(-a_n y^{2n})$$

where $a_n q^{2n}$ is the term with the highest exponent in the polynomial $V(q)$.

Note that

$$\beta^{1/2+1/(2n)} \frac{1}{2\pi} \int dp\, dx \exp\left\{-\beta\left[\frac{p^2}{2} + V(x)\right]\right\} \xrightarrow[\beta\to 0]{} \frac{1}{(2\pi)^{1/2}} \int dy \exp(-a_n y^{2n})$$

so that we may conclude

$$\lim_{\beta\to 0} \frac{\operatorname{Tr} \exp\{-\beta[\frac{1}{2}\pi^2 + V(q)]\}}{(1/2\pi)\int dp\, dx \exp\{-\beta[\frac{1}{2}p^2 + V(x)]\}} = 1$$

and

$$\lim_{\beta\to 0} \frac{\operatorname{Tr} \exp\{-\beta[\frac{1}{2}\pi^2 + \tilde{V}(q)]\}}{\operatorname{Tr} \exp\{-\beta[\frac{1}{2}\pi^2 + \hat{V}(q)]\}} = \frac{\int dy \exp[-(a_n - \epsilon)y^{2n}]}{\int dy \exp(-a_n y^{2n})} = \left(\frac{a_n}{a_n - \epsilon}\right)^{1/(2n)}$$

A discussion of the $\beta \to 0$ limit of path integrals in connection with the behavior of high-energy eigenvalues can be found in Ref. 44.

8. EQUIPARTITION BOUND ON THE ENERGY DENSITY

The bound $\beta\langle\phi(x)^{2n}\rangle \leqslant C$ derived in Sections 6 and 7 is essentially equivalent to an equipartition bound[10]

$$\langle H_\Lambda/|\Lambda|\rangle \leqslant CkT \tag{55}$$

for large T, uniformly in Λ. In other words, the energy per lattice site (i.e., per degree of freedom) is bounded by a multiple of kT.

To derive (55) we first recall the virial theorem. Let U_α be the unitary operator taking $\phi(x) \to \alpha\phi(x)$, $\pi(x) \to (1/\alpha)\pi(x)$. We have

$$H_\alpha = U_\alpha H U_\alpha^{-1} = \frac{1}{\alpha^2} \sum_{x\in\Lambda} \frac{\pi(x)^2}{2} + V(\alpha\phi(x)) + \alpha^2 \frac{1}{2} \sum_{\text{pairs}} J_{xy}[\phi(x) - \phi(y)]^2$$

Since $\operatorname{Tr}(e^{-\beta H_\alpha}) = \operatorname{Tr}(e^{-\beta H})$ we have, differentiating at $\alpha = 1$,[11]

$$-2\sum \left\langle \frac{\pi(x)^2}{2} \right\rangle + \sum \langle V'(\phi(x))\phi(x)\rangle + \sum J_{xy}\langle[\phi(x) - \phi(y)]^2\rangle = 0 \tag{56}$$

[10] This bound should not be confused with the infrared bound of Refs. 11 and 37, which applies only to the coupling term $\frac{1}{2}\sum_{\text{pairs}} J_{xy}[\phi(x) - \phi(y)]^2$ and not to the local terms.

[11] It is seen that the virial theorem is an example of a "local Ward identity" as defined in Section 3.

Using the equality (56), we may estimate $\beta\langle\pi(x)^2\rangle$ in terms of $\beta\langle\phi(x)^{2n}\rangle$, and of course $\beta\langle V(\phi(x))\rangle$ and $\beta\langle[\phi(x) - \phi(y)]^2\rangle$ can also be estimated in terms of $\beta\langle\phi(x)^{2n}\rangle$. Thus the estimates of Sections 6 and 7 lead, via the virial theorem, to the equipartition bound (55).

9. SPONTANEOUS SYMMETRY BREAKING FOR QUANTUM LATTICE FIELDS

We briefly review here the arguments leading to the existence of spontaneous symmetry breaking for multicomponent quantum lattice fields at nonzero temperature if the space dimension $s \geqslant 3$ (infrared bounds) and for one-component fields at nonzero temperature for $s = 2$ (Peierls argument).

9.1. Spontaneous Symmetry Breaking for Multicomponent Fields, $s \geqslant 3$

The lattice approximation to the path space gives a measure of the form

$$e^{-\mathscr{L}} \prod d\hat{\phi}_i$$

where

$$\mathscr{L} = \Delta\left\{\frac{1}{2} \sum_{\text{n.n. pairs}} \delta_{ij}^{-2}(\hat{\phi}_i - \hat{\phi}_j)^2 + \sum V(\hat{\phi}_i)\right\}$$

(n.n. = nearest neighbor) and where, if δ_α denotes the lattice spacing in the α direction ($\alpha = 0, 1,..., s$), $\Delta = \prod_\alpha \delta_\alpha$ and δ_{ij} is the δ_α associated with the direction of the bond between the pair i, j. The infrared bound becomes (see Refs. 10 and 37)

$$\frac{1}{|\Lambda|} \langle|\tilde{\phi}_p|^2\rangle \leqslant \frac{1}{2} \frac{1}{(2\pi)^d} \frac{1}{\epsilon(p)}$$

where

$$\epsilon(p) = \sum_{\alpha=0}^{s} \delta_\alpha^{-2}(1 - \cos p_\alpha\delta_\alpha), \qquad \langle\phi_x^2\rangle = \sum_{p\in\Lambda*} \Delta p \frac{1}{|\Lambda|} \langle|\tilde{\phi}_p|^2\rangle$$

Taking the limit $\delta_0 \to 0$, we obtain the path space for the quantum lattice field. Then $\epsilon(p)$ takes the form $(1/2)p_0^2 + \epsilon_s(\mathbf{p})$ and (recall that in the β direction the length is β) $p_0 = (2\pi/\beta)n$, $\epsilon_s(\mathbf{p}) = \sum_{\alpha=1}^{s} (1 - \cos p_\alpha\delta_\alpha)$. Taking the limit $V \nearrow \mathbb{Z}^s$ and assuming no long-range order, we obtain

$$\langle\phi_x^2\rangle \leqslant \frac{1}{2} \frac{1}{2\pi} \sum_{p_0} \Delta p_0 \frac{1}{(2\pi)^s} \int_{0\leqslant p_\alpha\leqslant 2\pi/\delta_\alpha} d^s\mathbf{p} \frac{1}{\frac{1}{2}p_0^2 + \epsilon_s(\mathbf{p})} \tag{57}$$

Note that, as $\beta \to \infty$, the right-hand side of (57) approaches ($s \geqslant 3$)

$$\frac{1}{2} \frac{1}{2\pi} \int_0^\infty dp_0 \frac{1}{(2\pi)^s} \int_{0\leqslant p_\alpha\leqslant 2\pi/\delta_\alpha} d^s\mathbf{p}\, [\tfrac{1}{2}p_0^2 + \epsilon_s(p)]^{-1} \equiv I \tag{58}$$

We may obtain a contradiction to (57) (thereby proving the existence of long-range order) by showing that $\lim_{\beta\to\infty}\langle\phi_x{}^2\rangle > I$.

A lower bound for $\langle\phi_x{}^2\rangle$ can be obtained from Bogoliubov's inequality

$$|\langle[C, A]\rangle|^2 \leqslant \beta\langle[C^+, [H, C]]\rangle\langle(AA^+ + A^+A)/2\rangle \tag{59}$$

taking $A = \phi_x$, $C = \pi_x$,

$$H = \sum_x \tfrac{1}{2}\pi_x{}^2 + V(\phi_x) + \tfrac{1}{2}\sum_{\text{n.n. pairs}} \delta^s[(\phi_x - \phi_y)/\delta]^2$$

(n.n. = nearest neighbor) where for simplicity we have taken $\delta_\alpha = \delta$, $\alpha = 1,\ldots, s$. We obtain

$$1 \leqslant \beta\langle V''(\phi_x) + 2s\delta^{s-2}\rangle\langle\phi(x)^2\rangle \tag{60}$$

If $V(\phi) = \lambda\phi^4 - \sigma\phi^2$, this gives

$$\langle\phi(x)^2\rangle \geqslant \frac{\sigma'}{12\lambda} + \left[\left(\frac{\sigma'}{12\lambda}\right)^2 + \frac{1}{12\lambda\beta}\right]^{1/2} \tag{61}$$

where $\sigma' = \sigma - s\delta^{s-2}$. (Note that as $\beta \to 0$ the lower bound goes like $\beta^{-1/2}$. The upper bound (Section 6) is also of this form.) In particular, $\langle\phi(x)^2\rangle \geqslant \sigma'/6\lambda$ for all β.

Choosing λ small enough or σ large enough so that $\sigma'/6\lambda > I$ [see Eq. (58)], it follows that there exists a critical β_c such that for all $\beta > \beta_c$ the quantum lattice field has long-range order.

The same methods work for ν-component fields. For example, if $V = \lambda\|\boldsymbol{\phi}\|^4 - \sigma\|\boldsymbol{\phi}\|^2$, taking $C = \pi^{(1)}(x)$, $A = \phi^{(1)}(x)$ in Eq. (59) gives

$$1 \leqslant \beta\left\langle 12\lambda\phi^{(1)}(x)^2 + 4\lambda\sum_{\alpha=2}^{\nu}\phi^{(\alpha)}(x)^2 - 2\sigma'\right\rangle\langle\phi^{(1)}(x)^2\rangle$$

By symmetry (we have no external field) $\langle\phi^{(\alpha)}(x)^2\rangle$ has the same value for all α. Then

$$\langle\phi^{(1)}(x)^2\rangle \geqslant \frac{\sigma'}{(8+4\nu)\lambda} + \left[\frac{\sigma'}{(8+4\nu)\lambda} + \frac{1}{(8+4\nu)\lambda\beta}\right]^{1/2}$$

The infrared bound (57) is now true for $\langle\phi^{(1)}(x)^2\rangle$ and thus the same method as previously used applies in the ν-component case.

9.2. Symmetry Breaking for One-Component Fields ($s = 2$)

The two-dimensional Peierls argument is used in the path space formalism. Let $\psi(x) = (1/\beta)\int_0^\beta dt\,\hat{\phi}(t, x)$. We follow the general discussion in Ref. 10. Define the characteristic functions $\chi_\pm(x) =$ char. fcn. of $\pm\psi(x) \geqslant 0$:

$$\chi_+(x) = \chi(\psi(x) \geqslant J) + \chi(0 \leqslant \psi(x) < J) = \chi_+^{(1)}(x) + \chi_+^{(2)}(x)$$
$$\chi_-(x) = \chi(\psi(x) \leqslant -J) + \chi(-J < \psi(x) \leqslant 0) = \chi_-^{(1)}(x) + \chi_-^{(2)}(x)$$

Then

$$\begin{aligned}\chi_+(x)\chi_-(x') &= \chi_+^{(1)}(x)\chi_-^{(1)}(x') + \chi_+^{(1)}(x)\chi_-^{(2)}(x') \\ &\quad + \chi_+^{(2)}(x)\chi_-^{(1)}(x') + \chi_+^{(2)}(x)\chi_-^{(2)}(x') \\ &= \sum_{\alpha=1}^{4} \chi_\alpha(x, x')\end{aligned}$$

The key ingredient in the Peierls argument is the estimate of a contour C which separates points where $\psi(x) > 0$ from points where $\psi(x) < 0$:

$$\mathrm{Prob}(C) = \left\langle \prod_{(x,x')\in C} \chi_+(x)\chi_-(x') \right\rangle$$

Using the chessboard estimate,[10,37] this probability is converted to an estimate of a pressure:

$$\mathrm{Prob}(C) \leqslant \sum_{\alpha(b)=1,2,3,4} \prod_{b\in C_r} \mathscr{P}_{\alpha(b)}$$

where b is a pair (x, x') in the reduced contour C_r. The pressures $\mathscr{P}_\alpha$ are estimated exactly as in the $P(\phi)_2$ models.[10,18]

10. ISING MODEL WITH TRANSVERSE FIELD

As discussed in Section 2.5, the Ising model with transverse field is a quantum system with path space equal to the continuum limit of a classical Ising model. We consider an anisotropic nearest neighbor model with finite-volume Hamiltonian for the region $V \subset Z^s$

$$-H_V = a \sum_{i\in V} \sigma_i^x + \sum_{\substack{\text{pairs}\\ i,j\in V}} J_{ij}\sigma_i^z\sigma_j^z + \sum_{i\in V} \sigma_i^z \Big(\sum_{j\in V^c} J_{ij}\omega_j \Big) \tag{62}$$

where ω_j are determined by the boundary condition in V^c (as discussed in Section 2.5) and

$$J_{ij} = \begin{cases} J_\alpha & \text{if } \text{i, j is a nearest neighbor pair in the } \alpha \text{ direction} \\ 0 & \text{if } i, j \text{ is not a nearest neighbor pair} \end{cases}$$

and $a, J_\alpha \geqslant 0$.

10.1. Ground State of the Quantum System

With $(+)$ boundary conditions $\omega_j = +1$ for all $j \in V^c$. The corresponding Hamiltonian is $H_V(+)$ with ground state $\Omega_V(+)$. The existence of dynamical instability (spontaneous magnetization) is indicated by

$$\lim_{V\nearrow\mathbb{Z}^s} (\Omega_V(+), \sigma_j^z\Omega_V(+)) = M \neq 0 \tag{63}$$

To estimate M we use results from the classical Ising model via the path space. (An exact solution of the quantum system in the one-dimensional case can be constructed by a transformation to fermion variables as in the two dimensional classical case.[31])

Consider the path space with mesh δ, which is a nearest neighbor Ising model with couplings $K_\alpha = \delta J_\alpha$ in the space direction α, and $K_0 = (a\delta)^*$ in the β direction (Section 2.5). We take the path space with length β in the β direction with $(+)$ boundary conditions: $\hat{\sigma}_i(\beta/2) = \hat{\sigma}_i(-\beta/2) = +1\ \forall i \in V$.

Let $\langle\cdots\rangle_{V+}^{\beta+}(\delta)$ denote the expectation with respect to this path space, which has $(+)$ boundary conditions outside all its boundaries. As discussed in Section 2.5,

$$\lim_{\delta\to 0}\langle\hat{\sigma}_i(0)\rangle_{V+}^{\beta+}(\delta) = \frac{\langle\boldsymbol{\sigma} = +|e^{-(\beta/2)H_V(+)}\sigma_i^z e^{-(\beta/2)H_V(+)}|\boldsymbol{\sigma} = +\rangle}{\langle\boldsymbol{\sigma} = +|e^{-\beta H_V(+)}|\boldsymbol{\sigma} = +\rangle} \tag{64}$$

which converges as $\beta \to \infty$ to

$$\lim_{\beta\to\infty}\lim_{\delta\to 0}\langle\hat{\sigma}_i(0)\rangle_{V+}^{\beta+}(\delta) = (\Omega_V(+), \sigma_i^z\Omega_V(+)) \tag{65}$$

Thus

$$\lim_{V\nearrow\mathbb{Z}^s}\ \lim_{\beta\to\infty}\ \lim_{\delta\to 0}\langle\hat{\sigma}_i(0)\rangle_{V+}^{\beta+}(\delta) = M \tag{66}$$

Equation (66) determines the spontaneous magnetization M of the quantum system in terms of classical Ising expectations.

By the Griffiths inequalities (Section 2.1a)

$$\langle\hat{\sigma}_i(0)\rangle_{V+}^{\beta+}(\delta) \searrow M_+^{(s+1)}(\delta) \qquad \text{as} \quad \beta \nearrow \infty,\ V \nearrow \mathbb{Z}^s \tag{67}$$

Thus

$$\langle\hat{\sigma}_i(0)\rangle_{V+}^{\beta+}(\delta) \geqslant M_+^{(s+1)}(\delta) \qquad \text{for all} \quad \beta,\ V \tag{68}$$

where $M_+^{(s+1)}(\delta)$ is the spontaneous magnetization for the infinite-volume, $(s+1)$-dimensional Ising model with couplings K_0, K_α as above. By a Griffiths inequality

$$M_+^{(s+1)}(\delta) \geqslant M_+^{(2)}(\delta) \tag{69}$$

where we have removed all couplings in the space directions except in the direction where K_α is maximum. Thus $M_+^{(2)}(\delta)$ is the spontaneous magnetization for the two-dimensional Ising model with couplings $K_0 = (a\delta)^*$ and $K_1 = \delta J_{\max}$ and is given by

$$M_+^{(2)}(\delta) = \left[1 - \frac{1}{x(\delta)^2}\right]^{1/8} \tag{70}$$

if $x(\delta) > 1$, where $x(\delta) = \sinh 2K_0 \sinh 2K_{\max}$. (See, for example, Ref. 34.) Note that in the limit $\delta \to 0$,

$$x(\delta) \to J_{\max}/a \tag{71}$$

Combining (66) and (68)–(71) we have immediately

$$M \geqslant \left[1 - \left(\frac{a}{J_{\max}}\right)^2\right]^{1/8} \tag{72}$$

if $a < J_{\max}$. We observe that for the one-dimensional quantum system, the exact solution[31] gives equality in (72).

To obtain results concerning the *absence* of spontaneous magnetization, we may apply Fisher's estimates for the Ising model.[9] Using estimates for self-avoiding walks, Fisher shows that for an Ising model with couplings $K_0, K_1, \ldots, K_s$ the magnetization of the system in an external field $h \sum \sigma_i$ satisfies

$$M(h) \leqslant (\tanh h) \frac{1 + x}{1 - x - 2(1 + x)y} \leqslant (\tanh h) \frac{2}{1 - x - 4y} \tag{73}$$

where $x = \tanh K_0 = e^{-2\delta a}$, $y = \sum_{\alpha=1}^{s} \tanh K_\alpha = \sum_{\alpha=1}^{s} \tanh \delta J_\alpha$. Thus if $a > 2 \sum_{\alpha=1}^{s} J_\alpha$, there exists a c and δ_0 such that for all $\delta < \delta_0$,

$$M(h, \delta) \leqslant c \tanh h \tag{74}$$

To apply (74) to the quantum system we take Dirichlet boundary conditions for the quantum system outside V and also for the path space at $\pm\beta/2$. This path space can also be regarded as arising from a certain boundary condition at $\pm\beta/2$ as discussed in Section 2.5. Again as $\beta \to \infty$ the expectations will converge to ground state expectations as in (65) [see Eq. (20)].

Thus we obtain

$$\langle \hat{\sigma}_j(0) \rangle_{V,\mathrm{D}}^{\beta;\mathrm{D}}(\delta) \leqslant c \tanh h \qquad \forall \delta < \delta_0$$

and therefore taking the limit $\delta \to 0$ and then $\beta \to \infty$, we obtain

$$(\Omega_V{}^{\mathrm{D}}, \sigma_j{}^z \sigma_V{}^{\mathrm{D}}) \leqslant c \tanh h$$

which gives

$$\lim_{V \nearrow \mathbb{Z}^s} (\Omega_V{}^{\mathrm{D}}, \sigma_j{}^z \Omega_V{}^{\mathrm{D}}) \leqslant c \tanh h \tag{75}$$

which shows that the spontaneous magnetization for the Dirichlet state is zero if $a > 2 \sum_{\alpha=1}^{s} J_\alpha$. A standard convexity argument shows that the spontaneous magnetization is zero for all translation-invariant boundary conditions [in particular for (+) boundary conditions]. A discussion is given in Appendix C.

Fisher's estimate is not optimal. For the two-dimensional Ising model[34]

$$\langle \sigma_0 \sigma_n \rangle \leqslant e^{-n\epsilon}$$

where

$$\epsilon = 2(K_1^* - K_2) = \ln \frac{\tanh \delta a}{\tanh \delta J} \xrightarrow[\delta \to 0]{} \ln \frac{a}{J}$$

and σ_0, σ_n are two spins separated in the space direction. This shows that the one-dimensional quantum model has exponential clustering if $a > J$. (For the exact solution see Ref. 31.)

10.2. Critical Temperature for the s-Dimensional Quantum System, $s \geqslant 2$

We discuss here the existence of a critical temperature for the s-dimensional quantum system, $s \geqslant 2$, which then implies the existence of a critical length for the $(s + 1)$-dimensional continuum Ising model.

Note first that at high enough temperature there is no spontaneous magnetization. This follows, since as a decreases, $K_0 = (a\delta)^*$ *increases*, and so by the Griffiths inequalities the spontaneous magnetization for the quantum model at temperature β is *smaller* than the spontaneous magnetization with $a = 0$, which is just the *classical* Ising model. The classical Ising model has zero spontaneous magnetization at high temperature by Griffiths' third inequality (see Section 3). Thus the existence of a critical temperature follows from the existence of a nonzero spontaneous magnetization for large β.

For $s = 2$, Ginibre has shown[14] (using a Peierls' argument with a different path space based on an alternative form of the Trotter product formula) that the two-dimensional quantum system does have a nonzero spontaneous magnetization for β large enough provided that a is small enough. By Griffiths inequalities, this implies the same result in higher dimensions. On the other hand, for $s \geqslant 3$ we may obtain an estimate for the critical temperature based on the infrared bounds[11] as discussed in Ref. 6.

Consider the quantum system in zero external field in a rectangular region with periodic boundary conditions and take $J_\alpha = J$ for all $\alpha = 1,..., s$. The Bruch–Falk inequality (see, for example, Ref. 6) gives a lower bound for the Duhamel two-point function:

$$f\left(\frac{\beta}{4} \frac{\langle [A^+, [H, A]] \rangle}{\frac{1}{2}\langle A^+A + AA^+ \rangle}\right) \leqslant \frac{(A^+, A)}{\frac{1}{2}\langle A^+A + AA^+ \rangle}$$

where $f(x)$ is the function defined by

$$xf(x) = [\tanh g(x)]^2$$

and $g(x)$ is the inverse function of $x = y \tanh y$: $y = g(x)$ (for $x, y \geqslant 0$). Note that $xf(x)$ takes any value between 0 and 1, $xf(x) \to 1$ as $x \to \infty$. Taking $A = \sigma_j^z$ gives

$$f(\tfrac{1}{4}\beta a\langle\sigma_j^x\rangle) \leqslant (\sigma_j^z, \sigma_j^z)$$

Since $f(x)$ is monotone decreasing in x,[6] and $0 \leqslant \langle\sigma_j^x\rangle \leqslant 1$, it follows that

$$f(\tfrac{1}{4}\beta a) \leqslant (\sigma_j^z, \sigma_j^z) \tag{76}$$

From infrared bounds (which follow because of the Ising path space), assuming no long-range order,

$$(\sigma_j^z, \sigma_j^z) \leqslant \frac{1}{2\beta J}\frac{1}{(2\pi)^s}\int_{0\leqslant p_i\leqslant 2\pi} d^s p\, \frac{1}{\sum_{\alpha=1}^{s}(1-\cos p_\alpha)} = \frac{I_s}{2\beta J} \tag{77}$$

Letting $x = \beta a/4$ and combining (76) with (77), we obtain

$$xf(x) \leqslant (a/J)I^s/8 \tag{78}$$

Now if $(a/J)I^s/8 < 1$ there exists an x_c, $0 < x_c < \infty$, such that

$$xf(x) > (a/J)I^s/8 \qquad \text{for all} \quad x > x_c$$

implying long-range order for all $\beta > \beta_c$. For $s = 3$, $I_s \approx \frac{1}{2}$, and this result would predict that the ground state would become degenerate if $a/J < 16$ (whereas we know from Section 10.1 that the ground state is degenerate if $a/J < 1$).

APPENDIX A. CORRELATION INEQUALITIES

We present here a proof of the generalized Griffiths inequalities for two-component rotators (Section 2.1c). We modify slightly the method of Ref. 1. By expanding the coupling term between different lattice sites in the usual way, the inequalities are proved by showing

$$\int (s+s')^a(s-s')^b(t'+t)^c(t'-t)^d\, d\mu_0(s,t)\, d\mu_0(s',t') \geqslant 0 \tag{A1}$$

for all nonnegative integers a, b, c, d. We have set $s = \sigma^{(1)}$ and $t = \sigma^{(2)}$ and s', t' are duplicate variables. (See, e.g., Refs. 1, 8, and 39.)

Given that

$$d\mu_0(s,t) = \exp[-V(s^2+t^2)]\, ds\, dt$$

we write

$$V(s^2+t^2) + V(s'^2+t'^2) = \mathscr{V}_e(s,s',t,t') + \mathscr{V}_o(s,s',t,t')$$

where

$$\mathscr{V}_e = \tfrac{1}{4}\{V(s^2 + t^2) + V(s'^2 + t^2) + V(s^2 + t'^2) + V(s'^2 + t'^2)\}$$
$$\mathscr{V}_o = \tfrac{1}{4}\{V(s^2 + t^2) - V(s'^2 + t^2) - V(s^2 + t'^2) + V(s'^2 + t'^2)\}$$

Note that $\mathscr{V}_e$ is invariant under interchange of s, s' and under interchange of t, t', whereas $\mathscr{V}_o$ changes sign.

The integral in (A1) is zero unless all a, b, c, d are even or all are odd, since μ_0 is invariant under $s \to -s$ and under $t \to -t$. The integral is non-negative if all a, b, c, d are even; thus we consider the case where all exponents are odd.

Let

$$I(\lambda) = \int (s + s')^{a+1}(s - s')^{b+1}(t' + t)^{c+1}(t' - t)^{d+1} \times \frac{1}{(s^2 - s'^2)(t'^2 - t^2)} e^{-\lambda\mathscr{V}_o} e^{-\mathscr{V}_e}\, ds\, ds'\, dt\, dt' \tag{A2}$$

Then $I(1)$ is the integral in (A1). If $\lambda = 0$, $I(0) = 0$ as follows by interchanging s, s'. The result $I(1) \geqslant 0$ then follows if we show $(d/d\lambda)I(\lambda) \geqslant 0$,

$$\frac{d}{d\lambda} I(\lambda) = \int (s + s')^{a+1}(s - s')^{b+1}(t' + t)^{c+1}(t' - t)^{d+1} \times \frac{\mathscr{V}_o}{(s^2 - s'^2)(t^2 - t'^2)} e^{-\lambda\mathscr{V}_o} e^{-\mathscr{V}_e}\, ds\, ds'\, dt\, dt' \tag{A3}$$

But

$$\frac{\mathscr{V}_o}{t^2 - t'^2} = \frac{1}{2}\frac{\partial}{\partial t^2}\bigg|_{t^2 = t_1^2} [V(s^2 + t^2) - V(s'^2 + t^2)]$$
$$= \frac{1}{2}[V'(s^2 + t_1^2) - V'(s'^2 + t_1^2)]$$

for some t_1^2 depending on t^2, t'^2. Then

$$\mathscr{V}_o/(s^2 - s'^2)(t^2 - t'^2) = \tfrac{1}{2}V''(s_1^2 + t_1^2) \geqslant 0$$

since $V(\xi)$ is convex for $\xi \geqslant 0$. (If V is not a smooth function consider first a smooth approximation to V and then take limits.)

This completes the proof.

We remark that the same method proves the GHS inequality of Section 2.1b for one-component spins. The method is related to the ones in Refs. 8 and 39. The GHS inequality follows from a Lebowitz inequality,[26] which again follows from inequality (A1), where now

$$d\mu_0(s, t) = d\nu(\sigma)\, d\nu(\tau), \qquad s = (\sigma + \tau)/2, \quad t = (\sigma - \tau)/2$$

τ is a duplicate variable for σ, and σ', τ' are reduplicated variables. Given that $d\nu(\sigma) = e^{-W(\sigma)}\,d\sigma$, we proceed as above with $V(s,t) = W(s+t) + W(s-t)$. Then

$$\begin{aligned}\frac{\mathscr{V}_0}{(s^2 - s'^2)(t^2 - t'^2)} &= \frac{1}{2(s^2 - s'^2)}\frac{1}{t^2 - t'^2}\\ &\quad\times \{[V(s,t) - V(s',t)]\\ &\quad\quad - [V(s,t') - V(s',t')]\}\\ &= \frac{1}{2(s^2 - s'^2)}\frac{1}{2t_1}[\partial_t V(s,t_1) - \partial_t V(s',t_1)]\\ &= \frac{1}{2(s^2 - s'^2)2t_1}[W'(s+t_1) - W'(s-t_1)\\ &\quad - W'(s'+t_1) + W'(s'-t_1)]\\ &= \frac{1}{8t_1 s_1}[W''(s_1+t_1) - W''(s_1-t_1)]\end{aligned}$$

and since W is even, we may suppose $s_1, t_1 \geqslant 0$. If $s_1 - t_1 \geqslant 0$ we use $(s_1 + t_1) - (s_1 - t_1) = 2t_1$ to obtain

$$\frac{\mathscr{V}_0}{(s^2 - s'^2)(t^2 - t'^2)} = \frac{1}{4s_1} W'''(\xi) \geqslant 0$$

since W' is convex for $\xi \geqslant 0$. If $s_1 - t_1 < 0$ we interchange s and t in this last step.

APPENDIX B. THE PATH SPACE FOR A SCHRÖDINGER PARTICLE

Let $H_0 = \frac{1}{2}\pi^2 + \frac{1}{2}m^2q^2 - \frac{1}{2}m$, the harmonic oscillator with frequency m. For H_0 explicit computation of the left-hand side of (2) shows[21] that the path space is given by the Gaussian variables $\hat{q}(t)$, $0 \leqslant t \leqslant \beta$, with covariance

$$\langle \hat{q}(t_1)\hat{q}(t_2)\rangle = \frac{1}{2\pi}\sum_K \Delta K \frac{e^{iK(t_1 - t_2)}}{K^2 + m^2} \tag{B1}$$

where $K = (2\pi/\beta)n$, $n \in \mathbb{Z}$, $\Delta K = 2\pi/\beta$. We may write

$$\hat{q}(t) = q_0 + \sqrt{\beta}\,r_m(t/\beta) \tag{B2}$$

where q_0, $r_m(t)$ are independent Gaussian processes with $\langle q_0{}^2\rangle = 1/\beta m^2$,

$$\langle r_m(t_1)r_m(t_2)\rangle = \sum_{n\neq 0} \frac{e^{i2\pi n(t_1 - t_2)}}{(2\pi n)^2 + m^2\beta^2}, \qquad t_1, t_2 \in [0, 1]$$

which is equal to

$$\frac{1}{4m\beta}(e^{-m\beta|x|} - e^{m\beta|x|}) + \frac{1}{4m\beta}\frac{e^{m\beta x} + e^{-m\beta x}}{\tanh(m\beta/2)} - \frac{1}{(m\beta)^2}$$

if $x = t_1 - t_2 \in [-\frac{1}{2}, \frac{1}{2}]$.

Now suppose $W(q)$ is a continuous function bounded from below. Then $H_0 + W$ is essentially self-adjoint (only a local L_2 condition is required for essential self-adjointness[23]). From the Trotter product formula

$$e^{-s(H_0 + W(q))} = \text{strong} \lim_{\delta \to 0} [e^{-\delta H_0} e^{-\delta W(q)}]^{s/\delta} \tag{B3}$$

The path space formula (2) gives

$$\begin{aligned}
&\frac{\operatorname{Tr}\{\exp(-\frac{1}{2}sH_0)\exp[-(\beta - s)(H_0 + W)]\exp(-\frac{1}{2}sH_0)\}}{\operatorname{Tr}[\exp(-\beta H_0)]} \\
&\quad = \lim_{\delta \to 0}\left\langle \prod_{K=0}^{K=(\beta - s)/\delta - 1} \exp\left[-\delta W\left(\hat{q}\left(\frac{s}{2} + K\delta\right)\right)\right]\right\rangle \\
&\quad = \left\langle \exp\left[-\int_{s/2}^{\beta - s/2} ds\, W(\hat{q}(s))\right]\right\rangle
\end{aligned} \tag{B4}$$

In deriving (B4) we have used the fact that the operators occurring in (B3) are uniformly bounded in norm and e^{-sH_0} is trace-class. For a discussion of the integral in (B4) see, for example, Ref. 25. We note simply that $\hat{q}(s)$ is continuous in s for almost every path, as follows by Kolmogorov's criterion.[41] From (B4) we see that

$$\begin{aligned}
&\lim_{s \to 0} \frac{\operatorname{Tr}\{\exp(-\frac{1}{2}sH_0)\exp[-(\beta - s)(H_0 + W)]\exp(-\frac{1}{2}sH_0)\}}{\operatorname{Tr}[\exp(-\beta H_0)]} \\
&\quad = \left\langle \exp\left[-\int_0^\beta W(\hat{q}(s))\, ds\right]\right\rangle
\end{aligned} \tag{B5}$$

It follows from (B5) that

$$\begin{aligned}
\frac{\operatorname{Tr}\exp[-\beta(H_0 + W)]}{\operatorname{Tr}\exp(-\beta H_0)} &\leqslant \left\langle \exp\left[-\int_0^\beta W(\hat{q}(s))\, ds\right]\right\rangle \\
&\leqslant \lim_{s \to 0} \frac{\operatorname{Tr}\exp[-(\beta - s)(H_0 + W)]}{\operatorname{Tr}\exp(-\beta H_0)}
\end{aligned} \tag{B6}$$

Since (B6) implies $\exp[-\beta(H_0 + W)]$ is trace-class (for any $\beta > 0$), the rhs of (B6) converges to $\operatorname{Tr}\exp[-\beta(H_0 + W)]$ and we have

$$\frac{\operatorname{Tr}\exp[-\beta(H_0 + W)]}{\operatorname{Tr}\exp(-\beta H_0)} = \left\langle \exp\left[-\int_0^\beta W(\hat{q}(s))\, ds\right]\right\rangle \tag{B7}$$

Evaluating $\operatorname{Tr}\exp(-\beta H_0)$ and using the representation (B2), we obtain from (B7)

$$\begin{aligned}\operatorname{Tr}\exp[-\beta(H_0 + W)] &= \left(\frac{\beta m^2}{2\pi}\right)^{1/2} \frac{1}{1 - \exp(-\beta m)} \\ &\times \int_{-\infty}^{\infty} dx \left(\exp -\frac{\beta m^2 x^2}{2}\right)\left\langle \exp\left[-\beta \int_0^1 dt\, W(x + \sqrt{\beta} r_m(t))\right]\right\rangle_{r_m}\end{aligned} \tag{B8}$$

Now let $W = V - \frac{1}{2}m^2 q^2 + \frac{1}{2}m$, where V is such that W is bounded from below for m sufficiently small: $V(x) \geqslant a + bx^2$ for some $b > 0$. Using $\int_0^1 r_m(t)\, dt = 0$ and

$$\left\langle \exp\left[\tfrac{1}{2}\beta^2 m^2 \int_0^1 r_m(t)^2\, dt\right]\right\rangle_{r_m} = [\exp(\tfrac{1}{2}\beta m) - \exp(-\tfrac{1}{2}\beta m)]/\beta m$$

we obtain

$$\begin{aligned}\operatorname{Tr}\exp\left[-\beta\left(\frac{\pi^2}{2} + V(q)\right)\right] &= \frac{1}{(2\pi\beta)^{1/2}} \int_{-\infty}^{\infty} dx \left\langle \exp\left[-\beta \int_0^1 dt\, V(x + \sqrt{\beta} r(t))\right]\right\rangle_r\end{aligned} \tag{B9}$$

where $r(t)$ is the Gaussian stochastic process with covariance

$$\langle r(t_1) r(t_2)\rangle = \sum_{n \neq 0} \frac{e^{i2\pi n(t_1 - t_2)}}{(2\pi n)^2}, \qquad t_1, t_2 \in [0, 1] \tag{B10}$$

which is equal to $\frac{1}{2}\{(t_1 - t_2)^2 - |t_1 - t_2| + \frac{1}{6}\}$ if $-\frac{1}{2} \leqslant t_1 - t_2 \leqslant \frac{1}{2}$.

The path space determined by (B10) is convenient because it is translation invariant ($t \to t + a$). To see the equivalence with the Wiener process, note that

$$\hat{r}(t) \equiv r(t) - r(0)$$

is the Wiener process for paths satisfying $\hat{r}(0) = 0 = \hat{r}(1)$.

In deriving correlation inequalities for quantum systems, an important role is played by the lattice approximation,[19] for which inequalities for classical lattice systems are applicable. It is easy to obtain an appropriate lattice approximation {not the usual one, which would involve $[\exp(-\delta\pi^2/2) \times \exp(-\delta V)]^{\beta/\delta}$} from Eqs. (B3) and (B5):

$$\begin{aligned}&\lim_{s\to 0}\lim_{\delta\to 0} \frac{\operatorname{Tr}\{[\exp(-\frac{1}{2}sH_0)][\exp(-\delta H_0)\exp(-\delta W)]^{(\beta - s)/\delta}\exp(-\frac{1}{2}sH_0)\}}{\operatorname{Tr}\exp(-\beta H_0)} \\ &\quad = \left\langle \exp\left[-\int_0^\beta W(\hat{q}(s))\, ds\right]\right\rangle\end{aligned} \tag{B11}$$

(and a similar formula for expectation values).

The kernel of $e^{-\delta H_0}$ is

$$\exp(-\delta H_0)(x, y) = \left[\pi\left(\frac{1 - \exp(-2m\delta)}{m}\right)\right]^{-1/2} \times \exp\left\{-\frac{1}{2}\frac{m}{\tanh m\delta}\left[x^2 + y^2 - \frac{2}{\cosh m\delta}xy\right]\right\}$$

and this leads to a nearest neighbor, ferromagnetic, pair-interacting, classical lattice model for which the correlation inequalities of Section 2.1 are valid. These inequalities then carry over to the limit (B11).

APPENDIX C. DEFINITION OF SPONTANEOUS MAGNETIZATION FOR THE ISING MODEL WITH TRANSVERSE FIELD

In this appendix we review standard convexity arguments to show that for a class of boundary conditions, the spontaneous magnetization for the Ising model with transverse field is independent of boundary conditions. We consider boundary conditions for the path space at $t = \pm\beta/2$ given by a measure $\mu(\boldsymbol{\omega}'', \boldsymbol{\omega}')$ on the configurations of the classical random variables $\hat{\boldsymbol{\sigma}}(\pm\beta/2)$.

Let us define a finite-volume pressure by the formula

$$P(\beta, V, \mu, \text{b.c.}) = \frac{1}{\beta|V|}\ln\sum_{\boldsymbol{\omega}'', \boldsymbol{\omega}'}\mu(\boldsymbol{\omega}'', \boldsymbol{\omega}')\langle\boldsymbol{\sigma}^z = \boldsymbol{\omega}''|e^{-\beta H_V(\text{b.c.})}|\boldsymbol{\sigma}^z = \boldsymbol{\omega}'\rangle$$

where μ is a measure on $\boldsymbol{\omega}''$, $\boldsymbol{\omega}'$ and b.c. denotes the boundary condition in V^c, the complement of the (space) volume V (see Section 2.5).

Since

$$\langle\boldsymbol{\sigma}^z = \boldsymbol{\omega}''|\exp[-\beta\hat{H}_V(\text{b.c.})]|\boldsymbol{\sigma}^z = \boldsymbol{\omega}'\rangle \xrightarrow[\beta\to\infty]{} \langle\boldsymbol{\sigma}^z = \boldsymbol{\omega}''|\Omega_V(\text{b.c.})\rangle\langle\Omega_V(\text{b.c.})|\boldsymbol{\sigma}^z = \boldsymbol{\omega}'\rangle \neq 0$$

$\{\Omega_V(\text{b.c.})$ is *strictly* positive since $\exp[-\beta H_V(\text{b.c.})]$ is positively improving[35]$\}$, it follows that

$$P(\beta, V, \mu, \text{b.c.}) \xrightarrow[\beta\to\infty]{} -E_V(\text{b.c.})/|V|$$

independent of μ, where $E_V(\text{b.c.})$ is the ground state energy of $H_V(\text{b.c.})$. Also,

$$\lim_{V\nearrow\mathbb{Z}^s} E_V(\text{b.c.})/|V| = \epsilon$$

independent of the boundary condition (b.c.) by standard arguments: If $V = V_1 + V_2$ and V_1 is decoupled from V_2 so that $H = H_{V1} + H_{V2}$, then

$E = E_{V1} + E_{V2}$. The coupling term in norm is small by a surface/volume factor yielding convergence of $E_V(\text{b.c.})/|V|$ and independence from boundary conditions (if $V \nearrow \mathbb{Z}^s$ in the sense of van Hove).

Thus

$$\lim_{V \nearrow \mathbb{Z}^s} \lim_{\beta \to \infty} P(\beta, V; \mu, \text{b.c.}) = \epsilon$$

independent of μ, b.c.

With an external field h in the z direction, the Hamiltonian has an additional term $-h \sum_{i \in V} \sigma_i^z$ and the path space formula shows that $P(\beta, V, \mu, \text{b.c.})$ is *convex* as a function of h, and therefore so is the limiting function ϵ. Furthermore,

$$\begin{aligned}
&\frac{\partial}{\partial h} P(\beta, V, \mu, \text{b.c.}) \\
&\quad = \frac{1}{|V|} \int_0^1 ds \sum \mu(\boldsymbol{\omega}'', \boldsymbol{\omega}') \Big\langle \boldsymbol{\sigma}^z = \boldsymbol{\omega}'' | \{\exp[-s\beta \hat{H}_V(\text{b.c.})]\} \\
&\qquad \times \sum_{i \in V} \sigma_i^z \exp[-(1 - s)\beta \hat{H}_V(\text{b.c.})] | \boldsymbol{\sigma}^z = \boldsymbol{\omega}' \Big\rangle \\
&\qquad \times \left[\sum \mu(\boldsymbol{\omega}'', \boldsymbol{\omega}') \langle \boldsymbol{\sigma}^z = \boldsymbol{\omega}'' | \exp[-\beta \hat{H}_V(\text{b.c.})] | \boldsymbol{\sigma}^z = \boldsymbol{\omega}' \rangle\right]^{-1} \\
&\quad \xrightarrow[\beta \to \infty]{} \frac{1}{|V|} \sum_{i \in V} \langle \Omega_V(\text{b.c.}) | \sigma_i^z | \Omega_V(\text{b.c.}) \rangle
\end{aligned}$$

independent of μ. Now if

$$\langle \Omega_V(\text{b.c.}) | \sigma_i^z | \Omega_V(\text{b.c.}) \rangle \xrightarrow[V \nearrow \mathbb{Z}^s]{} \langle \sigma^z \rangle_{\text{b.c.}}$$

independent of i [which is the case for Dirichlet and (+) boundary conditions, and for periodic boundary conditions with a nonzero external field], then it follows that

$$\lim_{V \nearrow \mathbb{Z}^s} \lim_{\beta \to \infty} \frac{\partial}{\partial h} P(\beta, V, \mu, \text{b.c.}) = \langle \sigma^z \rangle_{\text{b.c.}}$$

Since $P(\beta, V, \mu, \text{b.c.})$ is a convex function of h, it follows that tangents to $P(\beta, V, \mu, \text{b.c.})$ converge to tangents of ϵ. Then the spontaneous magnetization is given by $\lim_{h \downarrow 0} \langle \sigma^z \rangle_{\text{b.c.}}(h)$ and is equal to the right derivative of ϵ at $h = 0$ independent of the boundary condition (b.c.).

In fact for $\operatorname{Re} h > 0$ the Lee–Yang theorem together with Vitali's theorem can be used to show that $\epsilon(h)$ is analytic and therefore $\langle \sigma^z \rangle_{\text{b.c.}} = \partial\epsilon/\partial h$ independent of the boundary condition (b.c.) for $h > 0$.

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